THE PHILOSOPHY OF MATHEMATICS

SHAW

Mathematics Club
Wellesley College

Lectures on the Philosophy of Mathematics

Lectures on the Philosophy of Mathematics

By

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PREFACE

In the spring of 1915 the author was invited to deliver a course of lectures before a club of graduate students of the University of Illinois on the subject "The Philosophy of Mathematics." This club consisted of students who had had an ordinary college course in mathematics, for the most part. This fact tended to narrow the field to be covered, inasmuch as the more difficult questions of mathematical philosophy had to be omitted. It tended to widen the field in the way of making it intelligible to all students of fair mathematical knowledge, which could be accomplished best by considering mathematics constantly in its historical development. This class of readers is the one directly addressed in the lectures. The large class of secondary and collegiate teachers of mathematics is also addressed to a great extent.

The author cherishes the hope that the professional philosopher too may find some interest in these lectures, even though the more delicate problems are omitted or only referred to. If the student of philosophy finds enough mathematics here to characterize the field and give him a broad view over its hills and valleys, he will see it from the mathematician's point of view. Many have already accomplished this, notably, in America, Royce, and in France a whole school, as E. Boutroux, Brunschvicg, Milhaud, LeRoy, Winter, Dufumier, not to mention the philosopher mathematicians, Poincaré, P. Tannery, J. Tannery, Picard, Borel, P. Boutroux, and others. This view is necessary adequately to account for mathematics.

The object of the lectures is to consider the whole field of mathematics in a general way, so as to arrive at a clear understanding of exactly what mathematics undertakes to do and how far it accomplishes its purpose; to ascertain upon what presuppositions, if any, which are extra-mathematical, the mathematician depends. The references at the ends of the chapters will enable the student who desires to go into the topics treated farther than the discussions of the text permit, to make a start at least on such reading. They are not intended to be exhaustive, but merely suggestive. Students should consult constantly the *Encyclopédie des sciences mathématiques*, the Pascal *Repertorium*, and the *Taschenbuch* of Teubner.

The author has gathered his material from many sources, to all of which he acknowledges his indebtedness. The original source where possible is given for all quotations in order that the reader may find the original setting. It is hoped that notions due to other mathematicians have been in every case exactly expressed. Critics of the relative importance attached herein to many developments of mathematics are asked to keep in mind the purpose of the lectures. The synoptical table (pp. 196–97) is given as a suggestive guide to the text, and is doubtless incomplete in many ways.

James Byrnie Shaw

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CHAPTER I

MEANING OF THE PHILOSOPHY OF MATHEMATICS

When the thinking person of the present day stops to reflect upon the facts of the wireless telegraph and the long-distance telephone, not to mention many other just as important marks of human progress, and remembers in his thinking that the existence of the wireless telegraph is due to deductions of Maxwell by means of theorems that depend upon the square root of minus one, and that the possibility of the long-distance telephone depends upon investigations of Pupin by means of theorems that depend more directly upon the modern theory of expansions in fundamental functions, he appreciates to the full the power of this branch of human learning. When he further learns that the existence of conical refraction was pointed out to the physicist by a mathematician before it was discovered in a laboratory; that the existence of Neptune was pointed out to the astronomer before his telescope had noticed this wanderer in the remote heavens; when he learns that the mathematician by a theory related to the solution of the problem of finding the roots of an algebraic equation is able to say to the mineralogist "you will never find more than thirty-two distinct types of crystals"—when he meets such facts as these, he must invariably ask: "Who is this magician whose wand creates the marvelous and whose penetrating eye searches the hidden corners of the universe?" He may still listen with an amused smile to the curious properties of four-dimensional space, may delight in the

escape from the weary wastes of infinite space in a Riemannian finite universe, may be bewildered by the Minkowski imaginary-time axis, may exhaust his imagination in the vain effort to draw the crinkly curves, but he will not look upon all these as vagaries of a mystical dreamer. He will rather desire to seek permission of this Queen to enter her realm and explore it. "Conterminous[1] with space and coeval with time is the kingdom of Mathematics; within this range her dominion is supreme; otherwise than according to her order nothing can exist; in contradiction to her laws nothing takes place. On her mysterious scroll is to be found written for those who can read it that which has been, that which is, and that which is to come." He will wish to know what things belong thereto, the content of this division of human knowledge whose riches grow steadily richer year by year, while scientific theories are mined, assayed, and worked to exhaustion; while social orders are laboriously built up, serve their usefulness, and break down into ruins—this, the only permanent organization constructed by the human race. Indeed, he has for more than twenty-five centuries endeavored to account for this Antaeus, who acquires new strength whenever he touches the earth, yet whose towering form disappears to the average eye, in the shining clouds. The philosophers of the race have designed systems that charmed the mind for a while, resting their foundations upon the verities of mathematics; but they have turned out to be only temporary structures, occupying but a small part of the foundations of this ever-expanding temple of learning, which has become more solid while they have crumbled to pieces.

[1] Spottiswoode, *Report of the British Association for the Advancement of Science*, 1878, p. 31.

The corridors of this temple are many already, and it is no small task merely to walk through them; but it also has lofty towers from which are magnificent views over the whole range of the human intellect.

Not only will our speculative thinker desire to know the content of mathematics, but he will hunt for the central principles that control its evergreen growth. What does the mathematician seek to know? In the wilderness of lines and surfaces in geometry, in the puzzles of the realms of integers, on the slippery sands of the infinitesimal, in the desert of algebraic fields, what is it he is looking for? It is easy to understand the guiding principles of the natural sciences, but what are they in this science of the immaterial? Do these principles appear even in the small beginnings of mathematics, or have they emerged successively in the evolution of the race?

The thinker will further ask, What is the source of mathematical reality? Is it a dream which simulates the waking world so closely in parts that we are deceived into thinking that it is also real? Is there a Lobatchevskian space, or a Minkowski universe? Could a powerful telescope show us the antipodes, or could an electron wind its tortuous way according to a law expressed by the Weierstrass non-differentiable function? Where shall we trace the line all of whose points are at zero distance from each other, yet all distinct? What does a space with only seven points constituting its entirety mean for living existence, or what creature could have its being in a binary modular space? Is it true that most of the universes conceived in mathematics would be as sterile of life as the planets that swing in vacuous space? Is mathematics the subject[1] "in which we never

[1] Russell, *International Monthly*, 4 (1901), p. 84.

know what we are talking about, nor whether what we are saying is true?" Does mathematics rest on the granite of the earth, or on the mists of the air, or on the fancies of the poet? From Archimedes to Enriques, from Plato to Russell, from the Eleatics to Kant, from Heracleitos to Bergson, philosophers have endeavored to find an answer to these questions. Mathematics, at one period or another, has been a branch of physics, of philosophy, of psychology; but has Polignac's sentence[1] *Sophiae germana Mathesis* ever been surpassed?

In the Norse mythology there was an ash tree, Yggdrasil, which supported the universe. Its three roots were fast in the three realms where abode the shades of the dead, the race of mankind, and the frost-giants. Its lofty top was in the heavens, where abode the eagle of wisdom, and in the four corners gamboled four stags. We may well take this symbol as a very fair representation of mathematics—the sequoia that supports the universe of knowledge. It derives its stability from the roots that it sends out into the laws of nature, into the reasoning of men, into the accumulated learning of the dead. Its trunk and branches have been built during the past ages out of the fibers of logic; its foliage is in the atmosphere of abstraction; its inflorescence is the outburst of the living imagination. From its dizzy summit genius takes its flight, and in its wealth of verdure its devotees find an everlasting holiday.

Our speculative thinker, however, will desire to know something besides the content, the principles, and the reality of mathematics. He will ask what are the methods pursued in this field of investigation. In the laboratory

[1] Quoted by Cournot as the epigraph of the *Traité élémentaire de la théorie des fonctions et du calcul infinitésimal*, from Anti-Lucret. lib. iv, vs. 1083.

he sees glittering brass and nets of wires. Telescopes, microscopes, spectroscopes, balances, electrometers, all the paraphernalia of modern science are visible and have their obvious uses. But in the mathematical laboratory he will see perhaps a few curious models of surfaces or curves, or a few drawings, or a handful of instruments for computational purposes; but where is the apparatus for the discovery of the momentous facts and laws of this enormous field of learning? He may perchance watch the mathematician surreptitiously, as he sits with abstracted mien, his mental eye turned inward upon some intricate construction of symbols and formulae; he may even catch the flash of triumph when that eye sees the thread of the connection desired. Or he may build one of Jevons' logical machines and feed into it premises, terms, axioms, postulates, hieroglyphical symbols, and may endeavor to collect the stream of deductions that pour forth. Yet he will find that he has not accounted for the results of mathematics, that many of its finest flowers bloom without cause, much of its richest gold was not found by a prospector, or ground out in a stamp mill; that there is a spontaneity which eludes analysis, whose sudden outbursts are not the result of method. He will find that, while he may learn the way to prepare his intellectual fields, to remove the weeds, and to pulverize the soil, while he may plant the seeds, there may be no crop, or there may spring up strange and bewildering forms, as if some genie's touch had brought them forth from realms he knew not of.

After all these questions have been answered, we may then consider the right of this Queen[1] of all the sciences to

[1] Gauss, quoted by Sartorious von Waltershausen, *Gauss zum Gedächtniss* (1856), p. 79.

rule. Has mathematics a realm apart from human life, fitting daily experience in places closely enough to be of use, but still not at all identical with it; or is it, indeed, the very same as the realm of human life? Is the differential equation only a refinement upon the real law of physics, the irrational only an approximation to the actual number in nature? Is the universe stable or will it some day disappear, wind its way back into chaos, leaving nothing but the truths of mathematics still standing? Is it true that chance does not exist really but only in seeming, or is everything purely chance, and are the laws of the universe merely the curves which we have drawn through a random few of an infinitely compact set of points?

The consideration of these problems is what we mean by the philosophy of mathematics. If we can arrive at some answer, partial though it may be, it certainly would be worth while. Said Hilbert,[1] "Mathematical science is, in my opinion, an indivisible whole, an organism whose vitality is conditioned upon the connection of its parts." It may therefore be studied in the same way as any other organism, and such study of a branch of human knowledge is what we shall understand by philosophy.

We will therefore classify the problems we purpose to consider as follows:

1. The content of mathematics and its evolution
2. The central principles of mathematics
3. The source of the reality of mathematics
4. The methods of mathematics
5. The regions of validity of mathematics

In connection with each of the first four classes we shall find that definitions of mathematics have been given which were only partial. The varied character of its content

[1] Hilbert, *Bull. Amer. Math. Soc.* (2), 8 (1902–3), p. 478.

makes it impossible to reduce all mathematics to either arithmetic, geometry, logistic, calculus of operations, algebra, or transmutations. Nor can we define it any more successfully as a study of form, or of invariance, or of functionality, or of theory of equations. It has been the effort of some to define it in terms of its sources, but it is not possible to limit it to the phenomena of either the natural world, the world of universals, or the mental world. Definitions that are based upon its methods are no more successful, whether we emphasize its scientific procedure in observation, generalization, and analogy, or its intuitive insight into the nature of things, or its deductive chains whose invulnerable links support the weight of modern science; nor yet is it wholly the spontaneous product of the imagination of an artist whose material is not the solid granite of the architect, the flowing marble of the sculptor, the brilliant pigment of the painter, the rippling language of the poet, or the pulsing air of the musician, but is the delicate ether of pure thought. We cannot define it either by its applicability to the world at large, in any realm of validity. For[1] "it owes its true existence to a purely idealistic need, which, indeed, is akin to the need of a knowledge of nature, and to its satisfaction directly serviceable, but neither in it has its root, nor will ever be merged therein, ever so little." In none of these ways may it be defined, for it is all these and more. It may be studied in its various aspects just as man may be studied, but no definition based upon any one aspect will be sufficient to include the living creature itself. It may be defined only by being described, just as we may not define, but must describe, a giant sequoia. Mathematics becomes thus independent of any other

[1] Pringsheim, *Jahrb. Deutsch. Math. Ver.*, 13 (1904), p. 380.

branch of human knowledge. It is autonomous, and in itself must be sought its nature, its structure, its laws of being. Not in philosophy, not in science, not in psychology, not in logic, can we discover these things, but only in mathematics. It does not yield us transcendental space or time, or the categories of reason. It does not tell us whether physical space is Euclidean, Riemannian, or Lobatchevskian. It does not say whether only local time exists all over the physical universe. It does not decree the way of a cell of protoplasm or the logarithm of a sensation. It is itself a living thing, developing according to its own nature, and for its own ends, evolving through the centuries, yet leaving its records more imperishable than the creatures of geology. As such we will study it.

We will inspect the problems of the philosophy of mathematics a little more closely and thus will see their contents. We find that the subject-matter of mathematics can all be put into one or another of the divisions following (see table, p. 196):

STATIC MATHEMATICS	1. Numbers, leading to arithmetic 2. Figures, leading to geometry 3. Arrangements, leading to tactic 4. Propositions, leading to logistic
DYNAMIC MATHEMATICS	5. Operators, leading to operational calculus 6. Hypernumbers, leading to algebra 7. Processes, leading to transmutations 8. Systems, leading to general inference

These different divisions will be considered in some detail in the succeeding chapters. It is sufficient here to say that by numbers is meant, not only the domain of integers, but any general ensemble, at least in certain aspects. By figures is meant the aggregate of constructions in space,

whether of one kind or another, or of any number of dimensions. By arrangements is meant the study of signs and their combinations, arrangements, and other properties. By proposition is meant any assertion that has sense, whether true or false. By operators is meant such ensembles as arise by generating operations which act upon given base elements, producing cycles, groups, infinite series, or aggregations of these. By hypernumbers is meant such characters as that which makes unity differ from the square root of negative unity, or from a quaternion unit, or any unit of the hypercomplex number system. By process is meant any construction made up of the qualitative entities, and operations upon them. By system is meant any aggregation of mathematical entities which has sense, either consistent or inconsistent. These descriptions will become more definite in the detailed study of the divisions.

The central principles of mathematics can at present be classified under four heads, each of which appears in each of the eight divisions above, as follows:

1. Form, the particular character of constructions of any kind.

2. Invariance, the common characters of any class of entities.

3. Functionality, the correspondences of any class to any other class.

4. Ideality, the solutions of propositions of any kind. We will find that there are questions of form or structure entering each of the eight divisions of the subject-matter given above. Likewise in changes of form there will be in each case certain persistent characteristics which are the invariants of the forms in question. Forms of each kind will correspond to each other in one-to-one, many-to-one,

or many-to-many ways. And also in each division we shall find that the most important questions are reducible to finding the class of solutions of equations or propositions of given characters.

The sources of mathematical reality will be found to have been ascribed at various periods to four different worlds, as follows:

1. The natural world of phenomena, a reduction of mathematics to physics.

2. The world of universals, a reduction to logic.

3. The world of mental activity, a reduction to psychology.

4. The creative action of the intellect, a reduction to creative evolution.

Each of these ascriptions or reductions has, or has had, its earnest advocates, and naturally each contains some truth. No one is wholly true to the exclusion of the others, as we will try to show. Mathematics is not entirely a theory of space and time and number, of the nature of a theory of light, electricity, and magnetism. It is not merely the natural history of an existing[1] "inner world of pure thought, where all *entia* dwell, where is every type of order and manner of correlation and variety of relationship . . . . in this infinite ensemble of eternal verities whence, if there be one cosmos or many of them, each derives its character and mode of being . . . . there that the spirit of mathesis has its home and life." It is not the study simply of forms that the mind imposes upon the helpless universe of sense,[2] nor is it the study of the laws of thought.[3] Nor equally is the spirit of mathe-

[1] Keyser, *Hibbert Journal*, 3 (1904–5), p. 313.

[2] Kant, *Critique of Pure Reason*.

[3] Boole, *Laws of Thought*.

matics the goddess Athena, sprung from the head of Zeus—an astounding miracle in the universe of thought—but is rather spirit animating flesh.

The methods of mathematics are reducible to four, as follows:

1. Scientific, leading to generalizations of widening scope.
2. Intuitive, leading to an insight into subtler depths.
3. Deductive, leading to a permanent statement and rigorous form.
4. Inventive, leading to the ideal element and creation of new realms.

No one of these is used by any one mathematician to the exclusion of the others. A brilliant example is Poincaré, who says in his memoir[1] on "The Partial Differential Equations of Physics": "If one looks at the different problems of the integral calculus which arise naturally when he wishes to go deep into the different parts of physics, it is impossible not to be struck by the analogies existing. Whether it be electrostatics or electrodynamics, the propagation of heat, optics, elasticity, or hydrodynamics, we are led always to differential equations of the same family." Poincaré was a profound genius in his intuitive grasp of the essence of any problem he considered. His reasoning was, of course, strictly logical. And his creations in arithmetic invariants, asymptotic expansions, fundamental functions, double residues, and Fuchsian functions, are now classic. If one were in doubt as to the value of mathematics as a branch of study, he has but to consider these different ways in which a mathematician must think in his researches, in order to come to the conclusion that almost every power

[1] Poincaré, *Amer. Jour. Math.*, 12 (1890), p. 211.

of the mind is trained by the study of mathematics.[1] Perhaps no other study is so successful in developing particularly the power of invention, or had we better say, in stimulating the growth of the power of invention—that fruitful ability of the mind to bear new creatures of thought.

We now have before us a preliminary survey of the region we shall traverse, and the outlines of its main features. In connection with the traverse it will be necessary to consider some of the history of the development of these main features, for which it is well to refer to the standard texts.[2] There are several very suggestive addresses before various societies which will be useful for their different points of view; a partial list of these follows:

Royce, "The Sciences of the Ideal," *Science*, 20 (1904), pp. 449–462.

Bôcher, "The Fundamental Conceptions and Methods of Mathematics," *Bull. Amer. Math. Soc.* (2), 11 (1904), pp. 115–135.

Moore, "On the Foundations of Mathematics, *Bull. Amer. Math. Soc.*, 9 (1903), pp. 402–424.

Pringsheim, "Ueber Wert und angeblichen Unwert der Mathemetik," *Jahrb. Deutsch. Math. Ver.*, 13 (1904), pp. 357–382.

Picard, "On the Development of Mathematical Analysis and Its Relation to Other Sciences," *Bull. Amer. Math. Soc.*, 11 (1905), pp. 404–426.

Volterra, "Les mathématiques dans les sciences biologiques et sociales," *Revue du Mois*, 1 (1906), pp. 1–20.

Poincaré, "L'Avenir des mathématiques," *Bull. Amer. Math. Soc.*, 12 (1906), pp. 240–260.

[1] Keyser, *Science* (New Series), 35 (1912), p. 645.

[2] See particularly, *Encyclopédie des sciences mathématiques*.

Some others will be referred to in later chapters. In this connection it may be said that one should read the addresses of the presidents and other officers before such meetings as those of the American Association for the Advancement of Science, the American Mathematical Society, the British Association for the Advancement of Science, the London Mathematical Society, the various *Festschriften* in Germany, Académie des Sciences of France, the international congresses, and the like. In this way he becomes acquainted with the best thought of the mathematicians of the world regarding mathematics itself, and he will come to see it from every angle.

Books on the philosophy of mathematics are few. The latest is Brunschvicg's *Les Étapes de la philosophie mathématique.* The writings of Poincaré and of Picard on science should be consulted. A suggestive book is Winter's *Philosophie des mathématiques.* References may be found in these to a long series of articles of polemic nature in the *Revue de métaphysique et morale.* Many articles have appeared in the *Monist.* It is on the whole safe to say that the philosophy of mathematics has come into its own but recently and is at last disentangled from the snares of metaphysical discussions and the procedures of transcendental philosophy, and free to utilize all that has been done in the course of the ages, proceeding, however, henceforward on its independent path. The philosopher must, indeed, accept its results in his own general scheme and account for them. He may make its philosophy, as he has done in the course of history, the basis for his own system, but he cannot account for the past of mathematics on any basis but its own, nor can he predict its future.

CHAPTER II

NUMBER AND THE ARITHMETIZATION OF MATHEMATICS

The history of the development of the idea of number is one of the most instructive we meet in the study of mathematics. From the time of Babylon, and very likely long before, the common properties of integers were pretty well known. It must have been for a long time in that forgotten period of the world's history that men sought power over their fellows by secret knowledge, and particularly by knowledge of figures. We find, for example, the manuscript of Ahmes, dating from remotest times, entitled "Knowledge of All Dark Things." The Babylonians had tables of squares and cubes, some knowledge of progressions, rules for areas, and elementary knowledge of the circles of the stars. This knowledge had developed through the ages as man faced the world and its problems. The struggle for existence made number necessary, and the fortunes of dynasties made a study of the stars and what they could tell of the future a luxury that could only be bought from those who had leisure for intellectual effort. What arithmetic consisted of in the earliest times can only be conjectured from what we find now in the least enlightened tribes of the world. From the barter of goods—skins for bolts of calico or cocoanuts for glass beads—to the buying and selling of wheat in the pit is a long stretch and is possible only by the use of number.

Whoever first invented number was a genius of the highest order, just as was he who first invented language.

Words enable us to dispense with objects or their representations and yet to make use of objects in so far as words represent them. Numbers go farther than do mere words, for they enable us to refer to the distinction between objects, without the necessity of identifying the objects. To arrive at a number we do not perform an act of abstraction, as when we reach the general term "dog," for example. In this we ignore all the distinctions between different dogs and retain only the common characteristics that all dogs possess. But when we think of five dogs, for example, we are not thinking of the general term "dog," but of the individual, even though unidentified, dogs. We are enabled by the invention of the number 5 to keep the individuals distinct, and yet are not obliged to produce the descriptions of the individuals. Indeed, any five individuals would answer. Any two collections would be said to have the same number if they could be matched together exactly, individual to individual. A good example is that of the bank deposit and the safety deposit box. If one puts his money into the box, he draws out eventually the same coins. If he deposits his money in the bank, he draws out eventually coins which in all probability are not any of them the same that were deposited, but which yet have the same value. The bookkeeper's account is the controlling factor, and this is a numerical account. The invention of number enables the bookkeeper to control the actual money employed without even seeing any of it.

So far back does the invention of number go in its history that some have contended that it is not an invention of the intellect, but an innate endowment of it. Kronecker[1] says: "God made integers, all else is the

[1] Weber, *Jahrb. Deutsch. Math. Ver.*, 2 (1891–92), p. 19.

work of man." Yet the history of the evolution of the other kinds of numbers leads us to think that the integer is no exception to the others, but had its origin in that mysterious creative power of the intellect, which, we shall find, permeates all the history of mathematics.

We may consider that the first stage of development of number culminated in the school of Pythagoras, presumably about 500 B.C. This school of philosophy was a fraternity, as well as a school, and regarded the secrets of mathematics as too sacred for the uninitiated. They tried to reduce the universe to number, ascribing mystic powers to the different small integers. Thus 10 was a marvelous number because it included five odd numbers, and five even numbers, 5 itself being the mystic number of the pentagram; further, 10 included five primes, and five composites, and is the sum of an odd number, an even number, an even-odd number, and an even-even number. They divided numbers into perfect, abundant, and deficient. A perfect number is the sum of all its divisors exclusive of itself, an abundant number is less than the sum of its divisors, and a deficient number exceeds the sum of its divisors. For example, $12<1+2+3+4+6$, and is thus an abundant number, while $8>1+2+4$, and is a deficient number. The first perfect number is $6=1+2+3$, the next is $28=1+2+4+7+14$. It was shown in Euclid's time that if $2^{n+1}-1$ is prime, then the number $2^n(2^{n+1}-1)$ is perfect. This formula includes the two numbers 6 and 28, given above, the next number being 496. It is not known that there exist any perfect numbers which are odd, but, if there are such, several theorems have been proved to hold for them. The properties relating to the divisors of a number were also extended to two numbers. Two numbers were called

amicable if each was the sum of the divisors of the other, as, for example,

$$220=1+2+4+71+142,$$

and

$$284=1+2+4+5+10+11+20+22+44+55+110.$$

The Pythagorean fraternity represented numbers by groups of points, arranging them in various designs which showed certain properties. It was easy to see in this way that the sum of the first N odd numbers is N^2, since each odd number of points added will just suffice to border the adjacent sides and furnish a new corner for the square already constructed. So the sum of the first N even numbers is $N(N+1)$, called a heteromeque number, as may be seen easily by arranging the points in a rectangle whose length has one more row than its breadth. The successive addition of even numbers of points will preserve this shape. They invented also harmonic progression and called the cube the perfectly harmonious solid, since it had twelve edges, eight vertices, and six faces, and the numbers 12, 8, and 6 form a harmonic progression. Music was of course connected in this manner with mathematics.

On the basis of all these and such other mathematical properties as they were acquainted with, the Pythagorean fraternity announced that the universal principle of philosophy was mathematical harmony and proportion. The universe thus became entirely rational, contained an absolute and universal essence, and was above the accidents of sense. But one unlucky day, whose mystic number must have been peculiarly satanic, the fraternity discovered that the diagonal of a square and the side could not both be expressed by integers in any way.

The beauty of the universe was swallowed up in the inextricable confusion of the hopelessly irrational. The awful secret was divulged finally by a renegade member, who met a prompt and just punishment in the waves of the Mediterranean. However, the secret was out and mankind shouldered again its weary burden of explaining the universe.

In recent times there have been attempts to reinstate the integer as the only real number, all else being merely symbolic. Kronecker, for instance, endeavored to place the whole theory of fractions, irrationals, roots of algebraic equations, and other parts of mathematics on the integer as basis, by means of the introduction of congruences. For example, if we replace by 0 the parenthesis $(5x-3)$ wherever it occurs, and if we reduce all expressions containing x, and consisting at most of a polynomial divided by a polynomial, by means of this substitution of zero, we accomplish exactly the same thing as if we directly set $x=\frac{3}{5}$. By the use of the congruence we avoid defining what we mean by a fraction. The same method of procedure extended enables us to restate many things in algebra, it is quite true. But on the whole we have shifted the difficulty and not annihilated it. We find later, in the preface of the *Diophantische Approximationen* of Minkowski, the assertion, "Integers are the fountainhead of all mathematics." But in reply to these extremes we have the statement of Hobson:[1]

An extreme theory of arithmetization has been advocated by Kronecker. . . . . His ideal is that every theorem in analysis shall be stated as a relation between integral numbers only, the terminology involved in the use of negative, fractional, and irrational numbers being entirely removed. This

[1] *Theory of Functions of a Real Variable*, p. 21.

ideal, if it were possible to attain it, would amount to a reversal of the actual historical course which the science has pursued; for all actual progress has depended upon successive generalizations of the notion of number, although these generalizations are now regarded as ultimately dependent on the whole number for their foundation. The abandonment of the inestimable advantage of the formal use in analysis of the extensions of the notion of number could only be characterized as a species of mathematical Nihilism.

The second stage in the evolution of number was entered when fractions were invented. Sexagesimal fractions were in use by the Babylonian astronomers and survive in our degrees, minutes, and seconds. The Egyptians used unit fractions, which preserve the underlying notion of subdivision of a unit into equal parts. Some tables for the necessary reductions of combinations arising were computed and used. The Greeks, on the other hand, shunned fractions and did not have the idea of fraction as division or as ratio very clearly in evidence. The ratios of Euclid were actually orders of simultaneous arrangements of points along a line, and not the notion of fraction as we now have it. In fact, the actual distances between the ordered points might be arbitrary. For example, if we were to name a series of points as yellow points, for multiples of 2, another as blue points, for multiples of 3, and still another as green points, for multiples of 6, then the ratio of 2 to 3 would mean to Euclid an arrangement consisting of colored points, such as *green*, *yellow*, *blue*, *yellow*, *green*, *yellow*, *blue*, *yellow*, *green*, *yellow*, *blue*, *yellow* . . . . corresponding to the numbers 0, 2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18. . . . . We could just as well have made the colored points correspond to the numbers 0, 4, 6, 8, 12, 16, 18, 20, 24, 28, 30, 32, 36 . . . . the Euclidean ratio of

the numbers 4 and 6. Hence we declare that the two ratios, that is, the arrangements, are the same. The distances between the numbers have nothing to do with the order.

This notion of order was ingenious and leads to fruitful developments, but it is not the simple notion of ratio we have now, which has been much more fruitful. Indeed, the introduction of such a symbol as $\frac{3}{4}$, with the underlying idea that it implies, marks the entrance of the mind upon a new stage of development in its mathematical ability. The importance of this step, indeed, is fully as great as that of the invention of the Arabic notation for numbers. This creation of ratio gave the mind perfect freedom in carrying out division, which became always possible. Every number would divide every other number, for at this time zero was not yet existent.

It is easily evident that the list of integers is the same as the list of ratios whose numerators are exactly divisible by their denominators, and we are able to identify the class of integers (already existing) with this subclass of the new entities—the ratios—that we have created. This is a real identification, for we can easily conclude, by carefully considering the matter, that multiplication, as 3 by 4, for example, is only illustrated by the arrangement of four rows of three stars each, and does not consist in the arrangement, or in the addition of four 3's together; consequently division of 12 by 4 does not consist in the separation of 12 stars into 4 rows, or in the subtraction of 4 stars from 12 stars as many as 3 times with none left over. All this manipulation of objects may have led to the creation of number, addition, division, etc., but the concrete action merely furnishes the occasion for the use of the mathematics involved, and neither proves the result

nor does it define the mathematics. If one were to say, "Bring me a teacup," and the result of the request should be the possession of the teacup, the words used, their inflections, their order, are not proved by the teacup, nor does the action force one to use those terms. If one were not English, the result could be secured only by using a very different sentence. Language is an invention of the mind to enable man to react upon his fellows. So, too, mathematics is an invention of the mind, different from language, to enable it to handle its problems of existence. It is true that both have grown, that both are the result of circumstances to a considerable degree, that neither is purely arbitrary; but it is equally true that neither is an absolute ingredient of the external world, analyzed out by abstraction, that neither is an a priori ingredient of mentality, that neither arises from a world of universals. Each is the child of the spontaneity of the mind in its union with the natural world. Robinson Crusoe did not need a language, and a South Sea Islander needs very little, if any, mathematics.

The third stage in the development of number is the invention of the incommensurable. This has become eventually the irrational. In the incommensurable we deal with magnitudes which are compared with each other by the Euclid process, the same as the well-known process for finding the greatest common divisor of two integers. This process, when it comes to an end, leads to an expression of one magnitude in terms of the other by means of a continued fraction. If it does not come to an end, it leads to representation of one in terms of the other as an infinite continued fraction. In trying to find the measure of the diagonal of a rectangle in terms of the sides the Greeks soon came across the square roots of

numbers. The diagonal of a square has no integral nor fractional ratio to the side. Consequently this very simple entity already in the universe is not explainable in terms of integers, and the Pythagorean theory of the numerical structure of the universe breaks down, although much of the modern physics might be so construed as to restore this theory in natural science.

When we study the irrational number, the first consideration is the mode of representing an irrational. We can of course use particular symbols, as e, or π, or $\sqrt{2}$, but the supply would soon run short, not to speak of the impossibility of keeping such an enormous number of special symbols in mind or of ordering them in any way. The usual methods of representing the irrational consist of expressions of which the irrational is the limit. To accomplish this we set down a sequence of formulae, such that the general or nth member of the sequence can be found by some law of construction, the members of the sequence successively approaching closer to, and ultimately indefinitely close to, a limit, which limit is the irrational. Thus the sequence

$$1,\ 1+\tfrac{1}{2},\ 1+\tfrac{1}{2}+\tfrac{1}{6},\ 1+\tfrac{1}{2}+\tfrac{1}{6}+\tfrac{1}{24}\ \ldots$$

approaches the irrational number e, and the sequence of terms obtained by stopping at various places in the continued fraction

$$1+\frac{1}{2+}\,\frac{9}{2+}\,\frac{25}{2+}\,\frac{49}{2+}\cdots$$

is the irrational number $4/\pi$. A sequence of this kind may be determined by any numerical infinite series, that is, by successive additions; or by any infinite continued fraction; or by other iterated processes of combination

of numbers determined according to definite laws. We include in this list, of course, the expressions by means of definite integrals.

The invention of the sequence as an infinite series is due to Méray and was also later developed by Cantor, the notion of the latter being that the sequence itself was the irrational number. However, we must discriminate between the irrational number, which is an ideal invention of the mind, and the symbolism which enables us to identify the same irrational under various forms; just as we discriminate between the number 2 and the representation of 2 by figures or by the Roman II. The creation of the irrational with the invention of a machine which enables us to handle irrationals enables the mind effectively to carry out many processes that otherwise would be impossible; as, for example, finding the roots of algebraic or other equations, the calculation of transcendentals of various kinds, and so forth.

With the introduction of the sequence as a means of determining irrationals, began a process called the "arithmetization of mathematics." The object in view was that of the effectual reduction of all statements in mathematics to statements involving only rational numbers and sequences of rational numbers. It was proved that a sequence of irrational numbers which defined a limit could be replaced by a sequence of rational numbers defining the same limit, so that it seemed that an end to the extension of the notion of number had been reached in the irrational number. An attempt was made by DuBois-Reymond[1] to give back its place to the incommensurable, but little success followed the attempt. The natural outcome of the full arithmetization of

[1] *Die Allgemeine Functionentheorie*, 1882.

mathematics was the purely nominal character it gave the subject. Thus we find Helmholtz[1] asserting that "We may consider numbers to be a series of signs arbitrarily chosen, but to which apply a determined mode of regular succession or natural succession." If this were true, mathematics would become only an artificial game, played under certain arbitrary rules, and leading to no real truth.

The consideration of the class of irrationals, taken as a collection of individuals, brought forth a further mathematical creation—the ensemble. As a collection the class of irrationals is considered to be vastly more numerous than the class of rationals. Their density would not be sufficient to give us this notion, for, whether we consider rationals or irrationals, they seem equally dense. For between any two of either there is a third of the same kind; and, what is more to the point, there are members of the other set; that is, between any two rational numbers, however close, there are irrational numbers, and between any two irrational numbers there are rational numbers. Hence we are compelled from this point of view to think of the two as having the same density. But the rational numbers may be numbered by a rule which will account for each of them, though not in their natural order. To do this, we consider all the rational numbers the sum of whose numerator and denominator is a given integer N, each fraction being in its lowest terms, as, for example, if $N = 10$, we have the rationals $\frac{1}{9}$, $\frac{3}{7}$, $\frac{7}{3}$, $\frac{9}{1}$. It is evident that for a given N the number of rationals is finite, and that in following the successive numbers N we shall arrive somewhere at each and every rational $\frac{p}{q}$, at the farthest when

[1] *Zahlen und Messen*, 1887.

$N=p+q$, if not before. Now the sum of a finite number of numbers is finite, thus up to any given N the rationals can be numbered, or counted, and this will be true for any value of N, however large. It is obvious therefore that any given rational will have some integer assigned to it as its number in this method of ordering and that to each integer used in counting will correspond one and only one rational. If we endeavor to do the same thing for the irrationals, we find that it is not possible to devise a successful method of ordering the irrationals in this "denumerable" manner, since there are proofs that demonstrate that no such order can exist. The usual method of proof is to suppose that such order has been discovered and to show that inevitably some irrational would be left out, thus contradicting the hypothesis. This proof depends, however, upon the assumption that an irrational can be defined by a purely arbitrary assignment of an infinite set of coefficients. For instance, in one such proof it is supposed that the irrational is an incommensurable decimal, and that such a method of representation permits us to assign the successive figures for the decimal places purely at will to infinity—whatever that may mean. Whether this assumption can hold or not is an open question.

Indeed, the whole question of an infinite collection is brought to our notice by the irrational numbers and the ensemble they constitute. We meet here the problems of an actual infinity, which is quite different from the infinity of the calculus, the latter being no real infinity at all, but simply the possibility that a variable may take an unlimited set of values. We come face to face with the idea of continuity also, a notion which we find in our idea of magnitude and which we find we must attempt in the

arithmetization process to clutch in our sets of irrationals, no single set of which is in this sense continuous. We are led ultimately to define an arithmetical continuity, which is something quite different from the psychological continuity, but which we must make use of in the attempt to represent all mathematical problems in number form. The arithmetical continuity appears in the study of point-sets and in the notions of dense set, perfect set, and their extensions. These notions have little or no resemblance to the physical continuity of nature, or at least of our notion of nature. Instants are substituted for intervals, there is no duration properly so called, and such a conception as motion becomes a conception of the correspondence between numbers that represent position and another set of numbers that represent instants of time. The whole of modern analysis is founded upon such conceptions, and its ultimate utility is guaranteed by the success with which its methods have been applied to physical and geometrical problems. By introducing a "measure of a set" Lebesgue and others have found a means of handling sets satisfactorily.[1]

However, in the study of infinite collections we must enter several reservations. For example, such collections are defined as equivalent when a mode of correspondence between their members can be set up such that to each member of the one corresponds a definite single member of the other and vice versa. But, if the infinite collection is not given outright by some law, but grows by additions that depend upon the members already admitted, then it is not possible to compare two such collections, for one cannot be certain that the ordering up to any given stage must not be completely disarranged when further

[1] See De La Vallée-Poussin, *Les intégrales de Lebesgue*, 1916.

new members are admitted. In the foregoing example of the rational numbers the members of any subset belonging to the integer N can be found without knowing those of any other subset. Hence the argument of the possibility of the arrangement is not dependent upon its success at any stage. But in the arrangement of the incommensurable decimals against the integers the success we meet in constructing a decimal which has been left out is contingent upon a series of operations each of which demands that the entire set of decimals and integers be considered at each stage of the process. This is a manifest impossibility. It is clear that the incommensurable decimals can be arranged so that some of them are omitted, but so can the integers themselves. Thus the even numbers may be numbered, requiring all the integers to effect the numbering. Hence mere omission by a given scheme of some of an infinite collection does not prove that the collection is not equivalent to another collection on some other scheme of correspondence.

The attempt to introduce an actual infinity into mathematics has brought forth many paradoxes, all of them disappearing if the actual infinity is not introduced. These actual infinities have received the name of transfinites. They are inventions of the mind in direct extension of the notion of integer, so that classes consisting of an unlimited number of members might have numbers (called powers) assigned to them. The development of the theory is due to G. Cantor, to whose writings reference must be made. The criticisms on the theory may be found in the writings of H. Poincaré and related papers bearing upon the discussion which has been waged. Poincaré points out that, if we presuppose an external world of any kind, we probably will be forced to consider an

external and actual infinity as given. If, however, our notions of mathematics emanate from our own imagination, then an actual infinity is an impossibility.

Returning now to the point-set and leaving out of consideration the question of the cardinal number of the totality of the members of the point-set and its significance, we find that we have a very fruitful extension of the list of numbers. The notion of limit of a sequence is developed further, the various limits of all the sequences possible in a set of given numbers constituting the derived set, and when a set of numbers coincides with its first derivative set it is a perfect set. Now, a perfect set is amply sufficient to furnish the basis for the study of a continuity which is so far from the geometrical continuity that it may be as full of gaps as a sieve, yet which is so near the ordinary definition of continuity that no distinguishing feature is seen. For instance, in the modern treatments of the theory of functions of a real variable the theorems hold for a perfect set, as well as for the so-called linear continuum. Indeed, in the most recent work, functions are defined over an ensemble, and not over the linear continuum.[1]

Modern notions in physics indicate that the conceptions underlying the point-set theory may not be so far, as one might at first think, from the newer atomistic theories of physics.[2] Indeed, as Borel has pointed out,[3] the demands of physics have directed some of the great developments of mathematics in the past and may even now be forcing a new shoot to push its way forth.

[1] De La Vallée-Poussin, *Les intégrales de Lebesgue.*

[2] Van Vleck, *Bull. Amer. Math. Soc.*, 21 (1915), pp. 321–341.

[3] *Introduction géométrique à quelques théories physiques*, note vii.

The fifth stage in the development of the list of numbers we have just reached today. It is not content with the integer, the rational, the irrational, the point-set, but it demands a range of extremely general character. The numbers of this range are not necessarily ordered, as are those of a point-set, but may appear in geometric guise as lines, surfaces, and hypervarieties. For example, a loop of wire carrying a current will produce at any point of space a certain magnetic potential, which will be a function of the shape of the loop. Hence we must have as the independent variable the range consisting of the different loops that are possible in space of three dimensions. Such a range is called in general a functional space. The character of this new range is indicated by Hadamard[1] in the terms: "The functional space—that is to say the multiplicity obtained by varying continuously in any manner whatever possible—offers no simple image to the mind. Geometric intuition teaches us nothing a priori about it. We are forced to remedy this ignorance and can do so only analytically by creating a new chapter in the theory of ensembles which shall be consecrated to the functional continuum."

Such developments we find in the work of Fréchet[2] and Moore[3] and their students. The chief applications of such theory we find in the far-reaching developments of Volterra.[4] To these original papers we must refer those who wish to become further acquainted with the

[1] *L'Enseignement mathématique*, 14 (1913), pp. 1–18.

[2] *Math. Annalen*, 68 (1910), pp. 145–168; *Nouvelles Annales*, ser. (4), 8 (1908), 97–116, 289–317. Thèse.

[3] *New Haven Colloquium of the American Mathematical Society: An Introduction to a Form of General Analysis.*

[4] *Leçons sur les fonctions de lignes.*

functional space. For our present purpose it is simply sufficient to cite these investigations, in which physical intuition is helpless, to prove our general thesis that mathematics is a creation of the mind and is not due to the generalization of experiences or to their analysis; nor is it due to an innate form or mold which the mind compels experience to assume, but is the outcome of an evolution, the determining factors of which are the creative ability of the mind and the environment in which it finds the problems which it has to solve in some manner and to some degree. Every one of the different branches of mathematics will lead to the same conclusion, but in no case is the evidence more conclusive than in that of the field of number. When we find that the powerful methods of the infinitesimal calculus and its long career of successful solving of the problems of nature depend ultimately upon notions which in no way were derived from an analysis of the phenomena of nature or from an analysis of the nature of mind, we must admire all the more the ingenuity of the mind in devising such a sublime creation.

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CHAPTER III

SPACE AND THE GEOMETRIZATION OF MATHEMATICS

In the consideration of the problems of geometry we meet at the very outset the question of the reality of the visible and tangible universe. At first to question the existence of this startles us in the same way as the assertion of the motions of the earth with its "rock-ribbed hills and fertile valleys." Nothing seems more stationary as we look out over the stretch of plain or ocean than the earth. In a similar manner we are prone to believe that things are as we see them, or as we think they would be if we could come in contact with them. But the study of perspective drawing soon convinces us that objects surely do not have the shapes we see them in, for these alter with changes of position, and they even do not appear as they should in correct perspective. We learn that a hundred feet vertically does not look like a hundred horizontal feet, and that the sky reaches the ground many times farther away than it is above us at the zenith. Yet we find as the result of our endeavor to organize our daily experiences into a consistent and coherent whole that we have a notion of a real and permanent extension of three dimensions which we call empty space, and we assume that every real body must occupy more or less of this real space. In geometry we think we are studying the properties of this real space, which are not due to our physiological nor our psychological character, but are impersonal and have objective validity. The success which our deductions meet in innumerable predictions and calculations that

are afterward verified in the construction of engineering feats which are marvels of the ages leads us to believe that we have found eternal truths in the world of nature as imperishable as those of number. Geometry appeared to be that branch of applied mathematics which had invincible truth as its character, and which was the ideal toward which all science must strive. Systems of philosophy were founded upon it, and it was the pride of the intellectual world. Yet what a contrast between this height and the modern axiomatic treatment of geometry, in which almost any conceivable set of definitions which are not logically inconsistent, though they may sound absurd, may be used as the starting-point of a game called geometry, whose artificial rules and abstract situations have little to do with human experience apparently, or at most are convenient in the same sense that a meterstick is useful, or equally the king's arm, or the pace he sets. We assume objects A, B, etc., merely distinguishable from each other; for example, stars and daggers will do. We order them according to certain arbitrary rules. We set down the logical deductions therefrom, and we have a geometry on a postulational basis. Is this the last word and has reality vanished into vacuity and mathematics into a game of solitaire? Even at the fountain-head of much of this work we hear Klein[1] say: "I do in no wise share this view but consider it the death of all science: in my judgment the axioms of geometry are not arbitrary, but reasonable propositions which generally have their origin in space intuition and whose separate content and sequence is controlled by reasons of expediency."

[1] *Elem. math. vom höheren Standpunkte aus.*, Vol. 2 (1909), p. 384.

In addition we have latterly the troubles of the relativity theories, so that even physical space seems to be dissolving into dreams—we are making Alice-through-the-Looking-glass real. Space becomes time, and time space; things are when they are not, and words of ten syllables are easier than words of one syllable. These problems we will look at all too briefly in order to get some clear knowledge of the facts.

Four views have been held and are even at the present time held with regard to the nature of geometry. We are not referring here to metaphysical views of space and our notions of space. With these we have no concern, but will interest ourselves solely in the ideas of mathematicians.

One view is stated by Russell[1]: "All geometrical reasoning is, in the last resort, circular; if we start by assuming points, they can only be defined by the lines or planes which relate them; and if we start by assuming lines or planes, they can only be defined by the points through which they pass." This is the reduction of geometry to a system of logical deductions from a set of undefined elements and assumed postulates. A second view is stated by Bôcher:[2] "We must admit then that there is an independent science of geometry just as there is an independent science of physics, and that either of these may be treated by mathematical methods. Thus geometry becomes the simplest of the natural sciences, and its axioms are of the nature of physical laws, to be tested by experience and to be regarded as true only within the limits of the errors of observation." This is the

[1] *Foundations of Geometry* (1897), p. 120.

[2] *Bull. Amer. Math. Soc.* (2)2 (1904), p. 124.

reduction of geometry to a branch of physics. A third view is that of Poincaré,[1] "Geometry is not an experimental science; experience forms merely the occasion for our reflecting upon the geometrical ideas which pre-exist within us. But the occasion is necessary; if it did not exist, we should not reflect, and if our experiences were different, doubtless our reflections would also be different. Space is not a form of sensibility; it is an instrument which serves us not to represent things to ourselves but to reason upon things." This reduces geometry to a science of the ideas we have unconsciously stored up in our minds somewhere, and which are brought to light only when experience unlocks the door and takes down the shutters. A fourth view is that advanced by Halsted:[2] "Geometry is the science created to give understanding and mastery of the external relations of things; to make easy the explanation and description of such relation and the transmission of this mastery." This view reduces geometry to the study of the universal relations that hold between things. It leaves out of account the fact that things are not related of themselves, but that we do the relating. Likewise the statement of Poincaré assumes that all geometry lies latent now in every mind, Bôcher fails to account for the geometry that is not based upon experimental facts, and Russell denies virtually that there is any permanent truth in the conclusions of geometry. Each states an indubitable phase of geometry, but no one is complete in its statement of what geometry gives us. We ought rather to look upon geometry as the evolutionary product of the centuries, a dynamic rather than static view.

[1] "On the Foundations of Geometry," *Monist*, 9 (1899), p. 41.

[2] *Proc. Amer. Assoc. Adv. Sci.* (1904), p. 359.

Two and a half thousand years ago, and a century before the Pythagorean fraternity were endeavoring to reduce the universe to number, Thales of Miletus measured the heights of the pyramids by their shadows and predicted the solar eclipse of 585 B.C. Many of the propositions of the collection edited three centuries later by Euclid were known to him, and, if by mathematician we mean one who studies the subject for its own sake and not for predicting the fortunes of Chaldean kings, nor for increasing the wealth of the Nilean landowners, we may assert that the first mathematician was not only a geometer, but that his mathematics was intimately connected with its applications. Thales traveled in Egypt and no doubt was familiar with the empirical mathematics deduced by the Egyptians and used to build the pyramids and to fix the boundaries of the Nilean farms, and it is very significant that, while their rules deduced by observation were for the most part inaccurate, those deduced by Thales' intellect are still valid after the tests of twenty-five centuries. Indeed, no further argument is necessary to maintain the thesis that geometry is applied to the world of phenomena, but not deduced from it. On the other hand, it took a stretch of two and a half millenniums to reach the consideration of the foundations of geometry and the modern axiomatic systems of Pasch, Hilbert, Veronese, and others. This in itself is a sufficient answer to the contention that geometry is a collection of purely abstract axioms and the deductions from them under preassigned rules of logic. Geometry is neither a branch of applied mathematics nor is it deducible from purely logical constants. Nor is it true, as Mill asserted, that every theorem in geometry is a law of external nature, any more than it is true that every theorem of analytical mechanics

is a law of the natural world. In mechanics we are privileged to study forces that vary as the fifth power of the distance or inversely as the tenth power, but none such are known to exist in nature; and in geometry we prove many theorems that may be applicable to the natural world or may not be. Yet the truth in the theorems of geometry no one seriously contests, if he uses ordinary language. The brilliant Greek mind found here a most fascinating field for play, and groups of Greek dilletante gathered in the Athenian courts, drew diagrams in the sand, and argued over geometrical theorems, as well as the more fortunate who met in the Akadême over whose portal was the warning, "Let no one ignorant of geometry enter my door!" In such high repute was geometry held that Plato pronounced one day the immortal sentence, "God geometrizes eternally!"

Yet, on the other hand, we remember that the practical character of geometry makes the modern giant steel structure of many stories secure, and we agree with Clifford:[1] "Even at the time this book [Euclid] was written—shortly after the foundation of the Alexandrian Museum—Mathematics was no longer merely the ideal science of the Platonic School, but had started on her career of conquest over the whole world of phenomena."

For a thousand years geometry slowly developed from humble beginnings up to some knowledge of conics, while the history of the world was written in gorgeousness and misery, and the Roman eagles spread imperial rule and intellectual stagnation over the known world. This early period was closed with the tragic death at the hands of ignorant fanatics of the beautiful Hypatia, incarnation of Greek culture.

[1] *Lectures and Essays*, I (1901), p. 354.

More than a thousand years of intellectual night then dragged their somber way over the civilized world. Ignorance and superstition held an imperial sway worse than that of Rome, while the human mind gathered the energy that was to beat down the barriers that imprisoned it. Little by little during even another half millennium the new life of the spirit gathered force under the slime that had submerged it. America was discovered, printing was invented, and man began to attack nature to wrest her secrets from her. Early in the seventeenth century Descartes gave geometry a new start, from which time it has grown to be an enormous branch of the mathematical tree. He made it possible, indeed, for all mathematics to assume a geometric form, and we might say the geometrization of mathematics begins at that time. For example, in the shadows of an electric lamp we may see the theory of bilinear quadratics, and the nets of orthogonal curves on a surface contain the theory of functions of a complex variable. The many-faced crystal reflects in its facets the theory of groups, and in the dreams of imaginary four-dimensional space we have a perfect picture of electrodynamics, that is to say, certain differential equations. Even the theory of numbers finds in the geometric numbers of Minkowski a lattice-work for its progress. And on the physical side we find geometry so useful that it threatens to reduce physics to a study of the properties of certain constructions in space. Indeed it was Descartes who said, "Geometrical truths are in a way asymptotes to physical truths, that is to say, the latter approach indefinitely near without ever reaching them." And by keeping near to nature, as life has done in all stages of evolution, the geometry of Descartes's time burst out into a wealth of new forms.

The essence of the Cartesian geometry was the introduction of the manifold, that is to say, an entity consisting of a multiplex of two or more numbers instead of one number; or in the language of functions, a manifold is a range for functions of two or many variables. A point in a plane is a duplex of two variables, x and y; a surface is an equation containing three variables. It is easy to anticipate the generalization of Pluecker, which considers space to be a quadruplex of straight lines, or Lie's generalization, which makes space a quadruplex of spheres. An ensemble of lines dependent on one parameter forms a regulus; if dependent on two parameters, a congruence; if dependent on three parameters, a complex; and if dependent on four parameters, a space such as that in which we think we live. From this point of view any ensemble of geometric elements will define a space—there is no entity called space, or empty space. Further, in the early part of the last century Grassmann developed his science of space of N dimensions, meaning a point-space of N dimensions, that is, an ensemble of points which is dependent upon N parameters. Geometry thus became the science of manifolds, the elements constituting the manifolds being quite diversified. Indeed, to many mathematicians, geometry does not study space, but has become a language for analytic theorems on many variables. It thus adds to the notion of number the new notion, dimensionality, leading ultimately to an infinity of variables and functions of them. This is the geometrization of mathematics. It is expressed by Wilczynski[1] thus: "The invention of the analytic geometry has enabled us to state that every problem of analysis has a geometrical interpre-

[1] *Bull. Amer. Math. Soc.* (2), 19 (1912–13), p. 332.

tation and every problem of geometry may be formulated analytically."

But we do not escape all troubles by reducing geometry to the theory of manifolds. While, indeed, we may say that so far as points are concerned space demands three variables, so far as lines or spheres are concerned it demands four variables, and thus space is neither three- nor four-dimensional, we have a very important problem still to consider in the four-dimensional space of points. Treatises have appeared in no small number and with numerous theorems about the six regular hypersolids of four-dimensional point-space, about quadrics in the same kind of space, about the properties of knots in such space, and many other problems, and we may well ask the question: Is the space we live in really four-dimensional in points, and could it not happen that our fourth dimension is so small that we have never discovered we possess such a dimension? If space is really four-dimensional, how could we ascertain the fact? and what effect would it have on life? Not by motion clearly could we find four-dimensional point-space, for all the motions with which we are acquainted demand a three-dimensional space. If three points of a body are fixed, the body cannot move, whereas in four-dimensional space it could still rotate about the plane of the three points; as, for example, one might become like his reflection in a mirror. Then again, since we obviously have no intuitive knowledge of four-dimensional space, it clearly is not a product of the intuition. If a product neither of the study of the natural world nor of the intuition, it is left to be purely a mental affair, or one of the world of universals found in logic. But in either case at least the conception of a four-dimensional world of points is a direct creation of the

mind and not due to an analysis of our conceptions of the natural world. We are finally driven to the position that the hypergeometries are in reality creations of the mind, and that the world in which we live is actually a case of one of the different three-dimensional point-spaces about which we may prove theorems. The mind followed nature in developing the geometry of Euclid and in utilizing the more powerful methods of Descartes, but it finally burst into flowers of its own, and, once free from the trammels of experience, it may evolve according to its own nature. We find in the study of mathematics the science of these free creations of the mind in its endeavor to surmount the world of phenomena, some of them useful for the daily needs of humanity in its perpetual struggle to maintain its achieved elevation, many others produced as spontaneous acts, just as the musician plays, and the artist paints, simply for the pure love of creation. We may, indeed, say that we do not acquire, nor fall heir to, a ready-made space even in a physiological or a psychological sense, but that our space is a product of the intellect of each individual, elaborated day by day under the spontaneous action of the mind, though stimulated by the phenomena of our experience. Space is neither an external absolute whose laws we discover, nor is it a purely artificial game with which we amuse ourselves; it is the result of the living act of creation of the intellect. Poincaré points out, that if in vision the convergence of the eyeballs did not occur at the same time as the accommodation of the lens, but if the two could take place at will separately, then space would have seemed to us to be a four-dimensional point-space. But perhaps our struggle for existence on that basis would have led us to undertake things which would have been self-destructive to the race.

We may quote also from Brunschvicg:[1] "Space has its roots in experience, but it is achieved in the reason. The Intellect moves about in the world, yet it appertains to it to give itself a world. If we cast aside the fiction of a creation out of nothing, to which it is impossible to correlate either a distinct idea or a concrete image, the intellectual construction of space marks the highest degree of the creative power that man can conceive of or exercise."

We meet the same conclusions from another avenue of approach also, that is, in non-Euclidean geometry. Euclid had among his postulates one which read thus:

If a straight line meet two straight lines so as to make the two interior angles on the same side of it, taken together, less than two right angles, these straight lines, being continually produced, shall at length meet upon that side of the line on which are the angles which are less than two right angles. For ages mathematicians endeavored to deduce this postulate from the others and the axioms, with no success. Says Mach:[2] "It is a sublime spectacle which these men offer: laboring for centuries, from a sheer thirst for scientific elucidation, in quest of the hidden sources of a truth which no person of theory or of practice ever really doubted."

In 1733 an Italian priest, Saccheri, reduced the postulate to the statement that if $ABCD$ be such that angles A and B are right angles, and $AC=BD$, then the assumption that C and D are either acute or obtuse must lead to a contradiction. He showed that, if the postulate held for any one figure, it held for all. He thought he could refute the case of assumption that C and D were obtuse, but the assumption that they were acute gave him trouble.

[1] *Les étapes de la phil. math.*, p. 514.

[2] *Space and Geometry*, p. 115.

He really discovered (without being aware of it) the Lobatchevskian geometry.

Passing over other investigations, we find in 1829 the publication of Lobatchevsky and in 1833 that of Bolyai, in which it definitively appears that the postulate in question is not deducible from the others, and that, indeed, there is possible for space a geometry in which there are many lines through a point parallel to a given line, similar figures exist, the sum of the angles of a triangle is less than 180 degrees, the defect depending upon the area of the triangle, and trigonometry becomes the theory of hyperbolic functions. Said Halsted[1] with regard to Bolyai's small paper: "Bolyai's 'Science Absolute of Space'—the most extraordinary two dozen pages in the history of thought."

The intellectual world stood aghast as soon as the full import of these facts was known. It appears that the great Gauss had discovered the same results before, but had not the courage to publish them. It seemed that the structure that had been thousands of years building was tottering, and it had become a question of the laboratory or observatory to ascertain which of three space-worlds we inhabit. A huge triangle with vertices on three peaks in Germany was measured, and the resources of stellar measurements were taxed to solve the question. Even if for small figures it made no practical difference which geometry was the true one, if there were in all the sky a figure, for which it did make a difference, whose enormous sides could be traversed only in a century by the flash of light at a speed of 180,000 miles per second, yet the scientific world desired to settle the question, if possible. It is certain as the result of all such measurements

[1] *Introduction to Translation of Bolyai.*

that, if the difference can be discovered in this manner, it is less than the errors of observation of the present day.

Riemann in 1867 published a paper, which he had worked out some years before, in which he took the ground that our notions of space were of a general type, and that by experience we have learned that the geometry of space in nature should be, at least to a high degree of approximation, the Euclidean geometry. He also introduced a third geometry, which permits no parallels, nor similar figures, and which is finite, just as a sphere permits infinite motion, but is yet finite. It seems that no one previously had thought of this geometry which permits two lines to enclose a space, and which was the necessary complement of the Lobatchevskian. Indeed, the formulae of trigonometry in this geometry are the usual formulae of spherical trigonometry, if we make the argument or angle equal to the side of the triangle multiplied by the curvature, while in Lobatchevskian geometry we need corresponding hyperbolic functions. Whether we live in a three-dimensional point-space with a four-dimensional curvature which is positive, zero, or negative, is a question. Indeed, the curvature might not even be constant. As to whether we may ever settle the question, opinions differ. Poincaré took the position that the question had no sense. It was equivalent, he said, to asking which is the true measure for space, a yard or a meter, or whether rectangular or polar co-ordinates are the true co-ordinates for space.

The definitive conclusion, however, for our purposes is easily seen. If geometry is derived intuitively from experience, then we should know instinctively which is the geometry applicable to the world in which we live. If we have not yet ascertained the answer to this

fundamental question, then we do not derive our geometry intuitively. Neither does it come from a hypothetical world of universals, which themselves are derived from experience, like composite photographs, or even appear as invariants of experience. We are forced back again upon the conclusion that geometry is the direct creation of the human intellect, drawing its sustenance from the world of phenomena, but wonderfully transforming it, just as the plant transforms the water, the air, the carbon dioxide, into a flower. Kant had based his philosophy upon the objective certainty of Euclidean geometry and his philosophy had to go through a revision, for space was no longer a necessary form imposed upon the world when it took the clothing of the mind, but the mind was free to impose what form it liked. The transcendental character of Kant's philosophy went down into ruins, though his contention that the mind supplied its share to the content of experience was most astonishingly vindicated. Indeed, it turns out that without the creative co-operation of the mind there would be no experience. So great an importance is thus attached to the working of the mind that Bergson takes occasion to warn us that the intellect is merely one of the active agencies of life, whose products are produced for specific ends, but are not sufficient for all the ends of life.

Other researches also lead to the same conclusion. We need mention only the developments which were not of an analytical nature, although they may have been first suggested in that way.

This was the creation of projective geometry.[1] Of this Keyser[2] says: "Projective geometry—a boundless domain

[1] Monge (1746–1818)–Poncelet (1788–1868).

[2] *Columbia University Lectures* (1908), p. 2.

of countless fields where reals and imaginaries, finites and infinites, enter on equal terms, where the spirit delights in the artistic balance and symmetric interplay of a kind of conceptual and logical counterpoint, an enchanted realm where thought is double and flows in parallel streams."

We find, indeed, here the common ground for the union of all geometries, ordinary or N-dimensional, parabolic, hyperbolic, or elliptic. Starting from this foundation, we may be led to take the view of Klein and others that geometry essentially is only the theory of the invariants of different groups. For instance, the geometry of Euclid is the theory of the invariants of a certain group called the group of Euclidean movements, the ordinary group of translations and rotations. We find as another development the geometry of reciprocal radii, with such points of union as this: the geometry of reciprocal radii is equivalent to a projective geometry on a quadric properly chosen. We may study other groups, as that of rational transformations, indeed, all of Lie's (1842–99) transformation-group theory. We come back to the usual space again in the group of all continuous transformations, giving us analysis situs. We have thus finally created a very general geometry which may be illustrated as follows. If we were to undertake to study the geometry of the plane in its reflection in a very crooked and twisted mirror, we might not for a long time find out the usual theorems. But there would nevertheless be certain theorems that would remain true, however much distorted the image might be. This kind of geometry is very general and is independent of the Lobatchevskian, Euclidean, or Riemannian postulates. One feature of it, for example, is the three dimensionality of space.[1]

[1] Poincaré, *Revue de mét. et mor.*, 20 (1912), pp. 483–504.

We have from these investigations the definite result that even in a world of continual flux, where forms dissolve into others, point becoming point, or point becoming line, or point becoming circle or sphere, yet the intellect has created a mode of handling its problems of existence. We find, in other words, that an infinity of relativities are possible and of the most curious types, and even though the physicist is unable to locate any special point, line, plane, or configuration in space as an absolute point of departure, even though he must use changing scales of measurement, yet mathematics is superior to the world of sense and dominates it in all its forms. Whatever problems the ages may bring forth as to space or its measurement, or, indeed, as to its companion—time—we know to a certainty that mathematics will meet the situation, create a set of notions and relations sufficient to explain and manage the problems. If the Minkowski four-dimensional world of a mingled time and space becomes the most rational way to think of phenomena, we will find it just as easy as to think of the Copernican astronomy or the rotation of the earth.

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CHAPTER IV

ARRANGEMENTS AND MATHEMATICAL TACTIC

There is a charm for most persons in the arrangement of a group of objects in symmetrical designs. Three objects placed at the vertices of an equilateral triangle, or four at the vertices of a square, or three at the vertices of an equilateral triangle and one at the center, and other more complicated arrangements, which, however, preserve similar symmetries, appeal to the aesthetic sense as beautiful. The fact that in certain arrangements of objects under the action of physical forces we find them at the vertices of regular polygons, as in the experiments of J. J. Thomson on the arrangements of small magnets, or the arrangements of molecules in crystals, leads some philosophers to a view of the universe not very remote from that of Pythagoras, for the integer dominates these forms. The arrangement of the integers in various designs, such as squares, stars, crosses, and other forms, so that the sums of certain selected lines are all equal, has even been supposed to have magic power, and we find "magic squares" used as talismans against misfortune, and other mystic diagrams ascribed with great power. The study of magic squares has fascinated many persons, the underlying harmonies and mathematical laws furnishing the incentive to prolonged study. We may quote the statement of MacMahon:[1]

What was at first merely a practice of magicians and talisman makers has now for a long time become a serious study for

[1] *Proceedings of the Royal Institution of Great Britain*, 17 (1892), pp. 50–61.

mathematicians. Not that they have imagined that it would lead them to anything of solid advantage; but because the theory was seen to be fraught with difficulty, and it was considered possible that some new properties of numbers might be discovered which mathematicians could turn to account. This has, in fact, proved to be the case, for from a certain point of view the subject has been found to be algebraical rather than arithmetical and to be intimately connected with great departments of science, such as the infinitesimal calculus, the calculus of operations, and the theory of groups.

Likewise we find that certain games or puzzles have led to a considerable development of parts of mathematics. The problem of placing eight queens on a chessboard, for example, in such a manner that no one can capture any other one, has interesting connections, as well as a number of other chessboard arrangements. The 13-15-14 puzzle also turned out to be the source of some very interesting papers. The problem of putting together various chemical radicals in the possible combinations led Cayley to produce his memoirs on trees.

In all these different examples we find upon examination that the underlying question is that of the existence or the stability of proposed compounds or combinations of assigned elements under given rules of combination. We are thus led to the general problem of consistency and to the philosophy of mathematics which undertakes to reduce the subject to the determination of possible, that is consistent or co-existent entities. Indeed, from this point of view we could define mathematics as the subject whose problem was the determination of the ways of building up stable combinations of assigned elements under given rules called postulates. A book on the number of possible games of draughts or the openings of

chess would be just as much a mathematical book as one on the number and character of roots of an algebraic equation or the solutions of geometric problems. The mathematician, according to this view, spends his time in building castles of cards, satisfied if only they will stand up—possibly architecturally beautiful, we may grant, possibly much like real castles; but, after all, playing an artificial game, whose rules he may vary at his pleasure, reality and truth utterly absent from his play. Let us see if this is really true.

Suppose we consider the simple problem of arranging the 9 digits in a square form, so that the sum of those in any row or any column will be the same. First, since the sum of all is 45, the sum of each row or column must be 15. If, then, we break up 15 into three numbers in every way possible such that no number is repeated and none is above 9, we shall evidently arrive at all the possible rows or columns. This gives us the possible sets or partitions, as follows:

1, 5, 9; 1, 6, 8; 2, 4, 9; 2, 5, 8; 2, 6, 7; 3, 4, 8; 3, 5, 7; 4, 5, 6.

The columns and rows must be made of these combinations in some order and such that each number occurs exactly once. We may then set down the form

$$\begin{array}{ccc} 1 & 5 & 9 \\ 6 & & \\ 8 & & \end{array}$$

as the beginning of the possible arrays, since there are but two combinations containing 1, and neither can occur twice. In the second line there must be only such a combination as contains 6, and neither 5, 8, or 9. The only one is 2, 6, 7. If the 2 is put in the second column,

this column must be 2, 5, 8, but 8 has been used already. Thus 2 must go into the third column, and the array is now filled in only one possible way,

$$\begin{matrix} 1 & 5 & 9 \\ 6 & 7 & 2 \\ 8 & 3 & 4 \end{matrix}$$

To arrive at any other possible arrangement all we must do, and all we can do, is to permute the rows and columns of this form. In this problem we laid down arbitrarily certain conditions and selected arbitrarily a certain set of elements to combine under those conditions. By direct inspection we arrived at the only solution of the problem. In order to reach the different forms of the solution, we see that the sum of a row or of a column is invariant under a rearrangement of the rows or the columns. We could of course arrive at these separate forms independently of the notion of invariance of the sum by constructing directly each one of the forms in the same manner as the first was constructed, being careful to write each combination in every possible order and to select them systematically, so that no possible case be omitted. It is evident that two combinations above were not usable at all: 2, 5, 8, and 4, 5, 6. If we start with either, we find that we cannot possibly finish the array under the conditions laid down. This suggests that in any such problem there must be combinations that are excluded from use, and the investigation of the reason why they must be excluded would lead to very interesting theorems. Little has been done with such problems, however, and the reader is referred to books on magic squares[1] for particular solutions of various cases.

[1] Andrews, *Magic Squares and Cubes;* MacMahon, *Combinatory Analysis.*

A problem of a different type is that of the fifteen school girls. They are to walk in five triads on successive evenings, so that no pair of girls shall be together in a triad more than once. The question is, how many successive evenings can be arranged for? The answer is seven, every girl walking once exactly in company with every other. This problem has led to many investigations of the more general subject of triple systems. An exhaustive study of the possibilities is fairly difficult.

If we imagine given elements attached together by bonds, as, for example, in the chemical combination CH_3, which may be represented by the form or tree

$$\begin{array}{ccc} & \mathrm{H} & \\ & | & \\ \mathrm{H}- & \mathrm{C} & - \\ & | & \\ & \mathrm{H} & \end{array}$$

we may investigate what compounds of these elements or radicals can be made so that no bonds are left unattached. This is an important problem of organic chemistry. It led to very interesting developments by Cayley. It is clear that other forms could be constructed of a similar character and the possible combinations studied. We call such forms configurations.

A wider-reaching problem is that of the combinations and the arrangements of a set of objects. In its simplest form this is the study of permutations of n letters. For instance, with three letters, A, B, C, we can make six permutations. In these permutations there are cycles of letters, as, for example, we may have the two permutations ABC and BAC, AB constituting a cycle. The permutations may also have cycles of cycles. With a

large number of letters the varieties of combination become more intricate, leading to a very extensive theory.

In order to handle the problems of permutations, we have invented various notions, such as *cycle*, just mentioned; *transitivity*, which refers to sets of permutations in which each letter appears in each place once at least in some permutation; *imprimitivity*, in which the various permutations may be split up into combinations of simpler forms, these simpler forms appearing throughout as units, except that they may as combinations themselves be permuted. For example, the permutations $ABCD$, $BCDA$, $CDAB$, $DABC$, form a transitive primitive set, for each letter is first in one permutation at least, likewise second in one at least, third in one at least, and, finally, fourth in one at least. Again, AB, CD; BA, DC; CD, AB; DC, BA; is an imprimitive set; for each permutation is made up of combinations of the elementary cycles of AB and CD. We also find in the comparison of sets of permutations the powerful notion of *isomorphism* appearing. For instance, if we had the set $\alpha\beta$, $\gamma\delta$; $\beta\alpha$, $\delta\gamma$; $\gamma\delta$, $\alpha\beta$; $\delta\gamma$, $\beta\alpha$; we could say that it is isomorphic to the previously given last set. Thence we are led to the creation of the notion of *abstract set*, which would embody the characters of all isomorphic sets, without regard to their particular representations. We might, for example, consider all right triangles as isomorphic in certain theorems, and might thus consider that these theorems are really applicable to them because they are embodiments of a certain abstract form called *right-triangle*. This notion of isomorphism as a basis of generalization is widely used in mathematics. Its essence consists in considering the objects about which some proposition is stated to be isomorphic concrete cases of some abstract object of

which the proposition may be asserted. Indeed the world of mathematics consists to a large extent of these abstract objects. In the very beginning of arithmetic the individuals of a collection are considered to be isomorphic for the numerical properties to be considered. In algebra we might say the letters represent the abstract entity of which particular values of the letters are concrete cases.

In this manner various collections of elements may be set up and studied under diverse rules or postulates, leading to theorems which are universally valid because the conditions of dependence may be set up as we please. If later it is possible to find real objects in the sense of material objects, or objects of physics, chemistry, or other science, which for certain purposes may be viewed as isomorphic with our artificial set, then the theorems must be true of them also, since their materiality does not affect the basis of the original dependence or independence.

It is the recognition of these facts that enables the mathematician to interpret a system that he has worked out for a given set of objects with reference to a different set of objects. It becomes possible in this way to apply algebraic theorems to geometry and theorems of geometry to algebra. Indeed, to interpret a system which has been devised for a given set of elements with reference to a different set, is often extremely suggestive; for the possible new combinations of the second set may be very obvious, and, when these are interpreted back into terms of the first set we arrive at theorems that we might never have seen directly. This method of reciprocation of mathematical systems is one of the sources of the steady and enormous growth of the subject.[1] It is in this way that the consistency of the theorems of a Lobatchevskian

[1] Cf. Keyser, *The Human Worth of Rigorous Thinking*, chap. xiii.

geometry may be made isomorphic with the consistency of Euclidean geometry.

The analysis of existing systems into their ultimate elements and postulates becomes useful, too, when it enables us to correlate systems together as being in reality embodiments of one abstract theory. We quote Moore[1] on the subject: "The existence of analogies between central features of various theories implies the existence of a general theory which underlies the particular theories and unifies them with respect to those central features." We do not gain simply a knowledge of the foundations of mathematics in this analysis, but we find it easier to identify the foundations of very different systems and to see that they are aspects of one and the same abstract theory. It is for this reason that the study of many problems that look at first like mere puzzles or games to while away an idle hour may suggest methods of treatment of very difficult and important problems in other directions.

Besides the invariance in combinations that leads to the notion of abstract objects, we find correspondences between sets of combinations of different types which enable us to call one set a function of the other set. For instance, the set of hypernumbers which are the solutions of an algebraic polynomial is a function of a set of permutations on a given number of letters. This functionality was indeed the key that Galois found to unlock this difficult part of algebra. And the isomorphism between these permutations and the divisions of a sphere into triangles, enables Klein to present the solution of the quintic as a function of the division of a sphere into 120 equal triangles.

[1] *Introduction to a Form of General Analysis*, p. 1.

The discovery of isomorphisms and functionalities of this character requires the penetrating eye of a mathematical genius. After they are once brought to light, the less masterful can develop them and many others similar to them. The creation of the ideas that, like isomorphism and functionality, enable mathematics to attack more powerfully the world and its problems, requires the highest type of mathematical genius. The mathematician delights, it is quite true, in the harmonies of structure which he discovers in a game, but he is playing a game not so much for the pleasure of the game as for the suggestiveness of it. The game begets new mathematical conceptions. The point of view, however, that considers the whole of mathematics as a game, in which fantastic structures are built under arbitrary rules sees only a superficial phase of the activity of the mathematician. It completely fails to perceive the innate reality and permanence.

From this point of view the millenniums of Euclid's geometry have sufficed to build only a towering structure which, so far as we know, may yet, as further turrets and pinnacles are added to it, become unstable and go crashing down into the most hopeless ruins the ages have ever seen, for with it would go man's hope of ever really reaching any kind of certainty. Even though up to the present all physical reality has confirmed the truth of the theorems, yet these confirmations are only approximate, and the degree of approximation may grow less and less as time rolls on. From this point of view, that is true which stands today, although tomorrow it fall. No criterion of absolute truth, we are assured, exists. We may found our deductions on what premises we please, use whatever rules of logic we fancy, and can only know

that we have played a fruitless game when the whole system collapses—and there is no certainty that any system will not some day collapse!

We may easily be led into a metaphysical fog in this connection, in which we lose sight of the most refractory verity that life offers. "What is truth?" is an old question, and, when we see even the existence of the world, experience in any form, or the commonest fact, challenged to prove that it is not a dream, we need not be surprised to find mathematical facts thrown into the same state of unreality as the rest of the world. We may, however, safely say that the world, as we know it, is the only world we know, and to label it a dream is merely to call it by another name. To label it as unreal is to use the word reality in a new and strange sense. If terms are to retain their ordinary sense, then we have a perfectly definite problem before us. Here is mathematics, a structure of human activity which has gradually arisen through the ages. Its real existence no one can seriously deny. Our sole problem is to account for it so far as we can, and to decide as to its stability. Such a question as "Is it certain that 2 and 3 will always equal 5" is destitute of sense. So long as 2, 3, and 5, as well as the term addition, retain their present meaning, then $2+3=5$. If the term 2 comes to be refined or analyzed so that we may assert that there are two varieties of 2, then we might conceive that for one of these this proposition might need alteration. An example to the point is the term continuous function. It was once supposed that every continuous function was differentiable. Later it was shown that there were continuous functions that were not differentiable. To assert that the mathematician of the earlier period had made an error is to utterly misuse the term error. The earlier mathe-

matician had simply failed to analyze continuity. The continuous functions he was talking about *were* always differentiable. His definition of continuity, however, admitted a class of functions which had never entered any mathematician's mind until Weierstrass invented one. The fault was with the denotation of the term continuity and not with the theorem. The history of mathematics is full of similar cases. But still we hear the objector inquire as to how one knows that, even if 2 and 3 and 5 retain their meaning, he is certain that 2 and 3 will make 5 always. The definitive answer is that none of the notions of 2, 3, 5, or addition and equality, depend in any way upon time to determine them. That being true, the lapse of time cannot affect the proposition at all. We have a similar case in geometrical proofs. The proof is made from a particular figure and that figure is, indeed, drawn in color on some material. But none of these elements enter the proof. Not the particular features of the figure, the color of the crayon, nor the character of the material enter anywhere into the proof. Consequently, as they are not in any way parts of the result or the process, they cannot affect the conclusion.

Furthermore, the appeal to objects for the proof of a theorem or equally for the disproof of a theorem is an appeal in vain. "Not in me," saith the atom, or the molecule, or the block, "not in me is the straight line or the triangle, the number or the integral. No more than the stature of a man resides in the man do these reside in me. These things are to be found only in that replica of the world that the human creature has constructed for himself in his mind. These are things he applies to me to further his own purposes. They are none of mine." The measurement of the angles of countless millions of

triangles made of steel neither would prove that the sum of the angles of the triangle the mathematician is talking about is 180° nor would it disprove it. The reason is again that the theorem is independent of the object. If we prove a theorem as to a triangle or as to the number 2, the proof is nowhere dependent upon the material of an object, or upon its chemical constitution, or upon the day of the month, or upon the weather. This fact, which is obvious, is a sufficient reason for asserting that these theorems are therefore valid, irrespective of the material, or the time, or the weather.

A more subtle question is raised if we view these universal theorems as Enriques does, that is, as observed invariants of experience. If this is all they are, then, as experience proceeds through the ages, they may turn out ultimately to be only relative invariants, and might even be only approximately invariant. It is true that in a series of changes of form, in which we find an element that nevertheless does not change with the form, there is in this invariant element an independence from those features that accompany the changes. As an example, the harmonic ratio of four points is not disturbed by a projective transformation. But the ratio in question may be studied without considering it to be an invariant of such a transformation. We may study geometry, to be sure, from the projective point of view and reach the usual metrical theorems, but the more natural way to arrive at them is to study them directly. The fact that they may be looked at as invariants is a fact of which they are indeed independent. There is, so to speak, a higher degree of independence than that of invariance, namely, independence of an absolute type. This is the kind of independence that we will find in most mathematical

developments, if they are carefully analyzed. Consequently we can be certain that in studying those things which are independent in the absolute sense of time, place, and person, and not merely invariant as to time, place, and person, we really arrive at a permanent structure in the highest sense possible.

We find in the following a clear summary of the answer to the question:[1]

> The mathematical laws presuppose a very complex elaboration. They are not known exclusively either a priori or a posteriori, but are a creation of the mind; and this creation is not an arbitrary one, but, owing to the mind's resources, takes place with reference to experience and in view of it. Sometimes the mind starts with intuitions which it freely creates; sometimes, by a process of elimination, it gathers up the axioms it regards as most suitable for producing a harmonious development, one that is both simple and fertile. Thus mathematics is a voluntary and intelligent adaptation of thought to things, it represents the forms that will allow of qualitative diversity being surmounted, the moulds into which reality must enter in order to become as intelligible as possible.

It was C. S. Peirce[2] who defined Mathematics to be the "study of ideal constructions." He adds the remark: "The observations being upon objects of imagination merely, the discoveries of mathematics are susceptible of being rendered quite certain." The importance of viewing mathematics as a tremendous structure is brought out by this definition, from the humble magic square to vast systems such as projective geometry, functions of complex variables, theory of numbers, analysis in general. If

[1] E. Boutroux, *Natural Law in Science and Philosophy*, trans. by Rothwell, p. 40.

[2] *Century Dictionary; article "Mathematics."*

the mathematician were engaged only in ideal building, the definition might be sufficient. But we have already seen that he is interested in ranges and in multiple ranges, which may be considered to be the materials of building, as well as in the synthesis of these materials. And we shall see also that he is furthermore equally interested in the study of types of synthesis aside from the structures themselves. Like a master architect, he must study his stones and metals, he must design beautiful and useful structures, but he must do more. He must investigate the possible orders under various limitations. And, most of all, he is obliged to consider the actual processes of construction, which leads him into dynamic mathematics.

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Bragdon, *Projective Orament*, 1912.

CHAPTER V

LOGISTIC AND THE REDUCTION OF MATHEMATICS TO LOGIC

In the year 1901 we find in an article by Bertrand Russell:[1] "The nineteenth century, which prides itself upon the invention of steam and evolution, might have derived a more legitimate title to fame from the discovery of pure mathematics. . . . . One of the chiefest triumphs of modern mathematics consists in having discovered what mathematics really is. . . . . Pure mathematics was discovered by Boole in a work which he called *The Laws of Thought*. . . . . His work was concerned with formal logic, and this is the same thing as mathematics." Again, Russell says,[2] "The fact that all Mathematics is Symbolic Logic is one of the greatest discoveries of our age; and when this fact has been established, the remainder of the principles of Mathematics consists in the analysis of Symbolic Logic itself." Also in Keyser's address[3] we find: ". . . . the two great components of the critical movement, though distinct in origin and following separate paths, are found to converge at last in the thesis: "Symbolic Logic is Mathematics, Mathematics is Symbolic Logic, the Twain are One."

On the other hand, we find Poincaré[4] saying after his various successful attacks on logistic: "Logistic has to be made over, and one is none too sure of what can be saved. It is unnecessary to add that only Cantorism

[1] *International Monthly*, 4 (1901), pp. 83–101.

[2] *Principles of Mathematics*, p. 5.

[3] *Columbia University Lectures*.

[4] *Science et méthode*, p. 206.

and Logistic are meant, true mathematics, those which serve some useful purpose, may continue to develop according to their own principles without paying any attention to the tempests raging without them, and they will pursue step by step their accustomed conquests which are definitive and which they will never need to abandon."

What, then, is this logistic which made such extravagant claims in 1901 and in 1909 was dead? In order to understand it we must go back to the third century B.C., when Aristotle was developing the study usually called logic. The logic of Aristotle is well enough defined when it is called the logic of classes. A class may be defined in the following terms. Let us suppose that we start with a proposition about some individual, as for example, "8 is an even number," or as another case, "Washington crossed the Delaware." If, now, we remove the subject and substitute the variable x, we shall have the statements: "x is an even number, x crossed the Delaware," which are called propositional functions, from analogy to mathematical functions. In this case the functions have but one variable or undetermined term, x. If we let x run through any given range of objects, the resulting statements will be some true, some false, some senseless. Those that are true or false constitute a list of propositions. For example, we may say: "6 is an even number, 9 is an even number, this green apple is an even number," the first a true proposition, the second a false proposition, the third an absurdity. So I might say: "Washington crossed the Delaware, the Hessians crossed the Delaware, the North Pole crossed the Delaware," which are, respectively, true, false, and absurd, the first two cases being propositions. The propositional function with one vari-

able is called a concept.[1] The individuals that may be put into the empty term (which may be any word of the statement), the variable, and that yield true propositions, constitute the class of the concept. Thus the class of even numbers consists of a certain endless set or range of individuals; the class of presidents of the United States, a certain set of a few individuals; the president of the United States, of one individual; and the class of simple noncyclic groups of odd order may consist of no individuals at all. The individuals of a class may not be known; for instance, the daily temperatures at the North Pole or the odd perfect numbers. It is practically impossible to ascertain the individuals in the first class, and there may not be any in the second class mentioned. In case it can be shown that a class has no individuals it is called a null-class. It should be noted carefully that the individuals do not define the class, and the class only partly defines the individuals. The individuals define a *collection* not a *class*. The distinction is important. The same individuals may be referred to one or more classes. Nor is the relation of a member of a class to the class the same as the relation of a subclass to the class. For instance, we may discuss the class of numbers which either are multiples of 5 or give a remainder 1 when divided by 5, that is, 1, 5, 6, 10, 11, ... Now the class of fourth powers of integers are all either divisible by 5 or give 1 for remainder. Hence the fourth powers constitute a subclass of the first class mentioned, viz. 1, 16, 81, ... As a class, whatever is true of the larger class is true of the smaller. But of any fourth power as 81, say, we cannot assert that it has the property of divisibility by 5, and its relation to the class is different from the relation of the

[1] Couturat, *Encyc. of the Phil. Sci.*, Vol. I, p. 157.

subclass to the class. A subclass is said to be included in the class, not to be a member of it. This difference was first pointed out by Peano[1] and was not known to Aristotle. The two relations are indicated by the symbols ε and $\cdot(\cdot$, for instance,

Roosevelt ε President of the United States,
some square roots $\cdot(\cdot$ irrationals.

The symbol of a class is the inverted ε, ɜ, for instance,

$$x \ni \text{divisor of } 288,$$

read "the class of divisors of 288."

It is evident that a class is not a class of classes, for the latter is a class of propositional functions of one variable, the former a class of individuals.

Aristotle not only studied classes, with schemes for definition and subdivision of classes, but he introduced the syllogism as a means of reasoning. The syllogism is a succession of three statements of the inclusions of classes; in formal statement, Greek letters denoting classes,

$$\alpha\cdot(\cdot\beta,\ \beta\cdot(\cdot\gamma, \text{ then } \alpha\cdot(\cdot\gamma.$$

For example, Pascal's theorem is true of any conic, every circle is a conic, whence Pascal's theorem is true of every circle. For an individual circle we should have a different type of syllogism, a distinction not noted by Aristotle, namely,

$$\alpha\cdot(\cdot\beta,\ x \,\varepsilon\, \alpha, \text{ then } x \,\varepsilon\, \beta.$$

For instance, Pascal's theorem is true of circles, this figure is a circle, thence Pascal's theorem is true for this individual circle.

Logic rested with the Aristotelian development for many centuries and was supposed to be perfect. The

[1] *Rivista di matematica* (1891), p. 3.

regeneration of the subject has been ascribed to Leibniz, because he hoped to see a universal symbolism which would enable the complete determination of all the consequences of a given set of premises, to be easily carried out, just as mathematical formulae enable us to solve large classes of problems. This was his "Universal Characteristic." But it was reserved for a later day to bring to light the symbolic logic, and we may pass at once to Boole[1] and the nineteenth century. We shall find, however, in the invention of Boole and his successors, not the discovery of mathematics, but the mathematicizing of logic. The mind again devises new forms for its own use, new ideas by which to attack its problems.

Boole used letters to express classes, the conjunction of two letters indicating the largest common subclass; and the formal addition of two letters, the smallest common superclass. Then the six laws of logic are stated by the formal equations:

$$\begin{array}{cl}
a=aa & \text{(identity)},\\
a+ab=a,\ a(a+b)=a & \text{(absorption)},\\
ab=ba,\ a+b=b+a & \text{(commutation)},\\
aa=a,\ a+a=a & \text{(tautology)},\\
ab=aba,\ a=a(a+b) & \text{(simplification)},\\
a=ab,\ a=ac, \text{ then } a=abc & \text{(composition)}.
\end{array}$$

He introduced two constants called logical constants, represented by 1 and 0, with the meaning for 1, the minimum superclass of all classes considered, the logical universe; and for 0 the greatest common subclass of all classes, the null-class, or class of non-existents. It is understood that if a class is considered, the negative of the

[1] *The Mathematical Analysis of Logic* (1847); *An Investigation of the Laws of Thought* (1854).

class is also under consideration, represented by a'. If only one class is considered, then $1=a+a'$. If two are considered, $1=ab+ab'+a'b+a'b'$, etc. It is evident that

$$1a=a,\ 1+a=1,\ 0a=0,\ 0+a=a.$$

The invention of these notions which seem simple enough now was a great advance over the logic of Aristotle. It suggested, for example, the use of $1-a$ for a', with the formulae corresponding to algebra

$$a(1-a)=0,\ 1=a+(1-a),$$

which are the laws of contradiction and excluded middle. Any class may be dichotomized now in the form

$$x=ax+a'x=abx+ab'x+a'bx+a'b'x= \ .\ .\ .\ .$$

If x is a subclass of a, we indicate it by the equations

$$x=ax \text{ or } xa'=0.$$

The syllogism takes the very simple form

$$a=ab,\ b=bc,$$

then

$$a=abbc=abc=ac.$$

We have thus invented a simple algebra which, with the one principle of substitution of any expression for a letter which the letter formally equals and the reduction of all expressions by the laws of the algebra, enables us to solve easily all the questions of the older logic. Jevons[1] has stated the rule for doing this very simply: State all premises as null-classes, construct all necessary subclasses by dichotomy, erase all combinations annulled by the premises, and translate the remaining

[1] *Principles of Science; Pure Logic;* see also, *Studies in Deductive Logic;* also Couturat, *Algèbre de la logique* (*Algebra of Logic*), trans. by Robinson.

expressions, by condensation, into the simplest possible equivalent language.

Boole, however, made a further most important discovery: that there is a nearly perfect analogy between the calculus of classes and the calculus of propositions. That is, we may interpret the symbols used above as representing propositions, under the following conventions. If a is a proposition, a' is the contradictory proposition, ab a proposition equivalent to the joint assertion of a and b, $a+b$ the assertion of either a or b or both, 1 a proposition asserting one at least of all the propositions and their contradictories under consideration, and 0 a proposition asserting all the propositions and their contradictories simultaneously, that is, 1 asserts consistency, 0 inconsistency. A series of formal laws may now be written out and interpreted similar to those for classes. The syllogism, for instance, is the same,

$$a=ab,\ b=bc, \text{ then } a=ac;$$

or in equivalent forms,

$$ab'=0,\ bc'=0, \text{ then } ac'=0.$$

That is, if the assertion of a is equivalent to also asserting b, and if the assertion of b is equivalent to also asserting c, then the assertion of a is equivalent to the assertion of c. We may reduce the whole scheme of deduction as before to a system of terms which are the expansions of the possible list of simultaneous assertions, the premises annulling certain of these, and those remaining furnishing the conclusions. We should, however, note carefully that what we arrive at in this manner are not truths or falsehoods, but consistencies and inconsistencies. That is to say, we do not prove anything to be true or false

by the logic of propositions, we merely exhibit the assertions or classes with which it is consistent or compatible, or the reverse. In this sense only does logic furnish proof. It is obvious, however, that many new combinations of the symbols used are possible by these methods, and thus it is easy to ascertain the consistency of assertions that would not otherwise occur to us. While the premises evidently are the source of the conclusions, the conclusions are not the premises, and, on the one hand, the transition from the one to the other is made most easily by these methods, and the conclusions are new propositions consistent with the premises. A simple example will show what is meant: If a implies a', then a is 0; for, if $aa=0$, at once $a=0$. Conversely: if $a'a'=0$, $a'=0$, $a=1$. That is, a proposition which implies its contradictory is not consistent.

It should be noted that the calculus of propositions is not wholly parallel to the calculus of classes. This is shown particularly in the applications of a certain axiom, as follows: $(a\varepsilon \text{ true})=a\ Ax$. $a'=(a'\varepsilon \text{ true})=(a\varepsilon f)$. This is absurd for the logic of classes, since $a=1$ is a proposition not reducible to a class.

A useful form for implication is

$$(a \text{ implies } b)=(a'+b=1).$$

The next advance was due to C. S. Peirce,[1] who devised the logic of relatives, in which the propositional function with two variables appears, and which may readily be generalized into the propositional function with any number of variables,[2] giving binary, ternary, and then n-ary relatives. As simple examples we may omit

[1] *Mem. Amer. Acad. Arts and Sciences* (New Series), 9 (1870), pp. 317–378.

[2] Couturat, *Encyc. of the Phil. Sci.*, Vol. I, p. 170.

individuals that satisfy the proposition: A is the center of the circle c, arriving at the propositional function x is the center of y; or another example with four variables is found in: x is the harmonic of y as to u and v. The calculus of the logic of relations is obviously much more complicated than the previously known forms of symbolic logic. While some of the theorems and methods of the calculus of classes and propositions may be carried over to the calculus of relations, there are radical differences. Thus the relation xRy is the converse of the relation $y\breve{R}x$. These two relations are not identical unless R is *symmetric*. Again from xRy, yRz, we can infer xRz only if R is *transitive*. The ranges of a relation are the sets of individuals that satisfy the propositional function, when inserted for some one of the variables. The most complete development of these notions is to be found in Whitehead and Russell's *Principia Mathematica*. In the intoxication of the moment it was these outbursts of the mind that led Russell into the extravagant assertions he made in 1901. In the *Principia* there are no such claims. It should be noted, too, that the work of Whitehead in his *Universal Algebra* (1898) contained a considerable exposition of symbolic logic.

As soon as the expansion of logic had taken place, Peano undertook to reduce the different branches of mathematics to their foundations and subsequent logical order, the results appearing in his *Formulario*, now in its fifth edition. In the *Principia* the aim is more ambitious, namely, to deduce the whole of mathematics from the undefined or assumed logical constants set forth in the beginning. We must now consider in a little detail this ambitious program and its outcome.

The basal ideas of logistic are to be found in the works of Frege, but in such form that they remained buried till

discovered by Russell, after he himself had arrived at the invention of the same ideas independently. The fundamental idea is that of the notion of function extended to propositions. A propositional function is one in which certain of the words have been replaced by variables or blanks into which any individuals may be fitted. This isolation of the functionality of an assertion from the particular terms to which it is applied is a distinctly mathematical procedure and is entirely in line with the idea of function as used in mathematics. It enabled us above to define concept and relation, in a way, and it further makes quite clear in how great a degree mathematical theorems refer to propositional functions and not to individuals. For instance, the statement, "If a triangle has a right angle it may be inscribed in a semicircle," merely means right-angled-triangularity as a property is inconsistent with non-inscribability-in-a-semicircle as a property. In this mode of statement it is apparent to everyone that a large part of mathematics is concerned with the determination of such consistencies or inconsistencies. That it is not wholly concerned with them, however, is also quite apparent. For example, the calculation of π can be called a determination of the figures consistent with certain decimal positions only by a violent straining of the English language. And again, the determination of the roots of an equation is a determination of the individuals which will satisfy a given propositional function, and not a determination of the other functions consistent or inconsistent with that first function. There is a difference, well known to any mathematician, between the theory of the roots of a quadratic equation and the properties of quadratic functions of x. Again, the analysis of the characteristics of a given ensemble

is not a determination of the essential constituents of the propositional function whose roots are the individuals of the ensemble. Operators considered as such are not propositional functions, and neither are hypernumbers. It has been made quite clear, we hope, in what precedes, that much of the mathematician's work consists in building up constructions and in determining their characteristics, and not in considering the functions of which such constructions might be roots. There is a difference between the two assertions "$2+3=5$" and "If 2 is a number, and if 3 is a number, and if 2 and 3 be added, then we shall produce a number which is 5." We find the difference well marked in the logistic deduction of the numbers 1 and 2. The deduction is as follows:

Let us consider the propositional functions "$x\varepsilon\phi_1$ has only roots such that they cannot be distinguished," as likewise $x\varepsilon\phi_2$. . . . . For instance, let $(\ )=6$, the roots are $4+2$, 2×3, $12/2$, ... which are all indistinguishable in this propositional function. So also $(\ )=9$, $(\ )=4/3$. . . . . Then, if we call these propositions *similar* in that each *has indistinguishable roots*, we may consider next the propositional function p *sim* $[(\ )=6]$, where p is a variable proposition, which, however, is distinguished by the character of indistinguishable roots. We may now define the number 1 as the functionality in this functional proposition. That is to say, 1 is a property of propositional functions, namely, that of uniqueness in their roots. In mathematical language we might say: The character which is common to all equations of the form $(x-a)^n=0$, is called 1, thus defining 1. Now, while it is true, perhaps, that to seize upon equations with one root as cases in which oneness appears is a valid way to arrive at 1, nevertheless it is not at all

different from any other case in which oneness occurs, as in selecting one pencil from a pile of pencils. In a like manner 2 is defined as the common property of propositional functions which are relations of a twofold valence, that is, admit two series of roots, the series in each case consisting of indistinguishable individuals. The truth of the matter is that the definitions given are merely statements in symbolic form of cases in which the number 1 or the number 2 appears. The two numbers have in nowise been deduced any more than a prestidigitateur produces a rabbit from an empty hat, but they have first been caught, then simply exhibited in an iron cage. The fact that functions are useful things we cheerfully admit, but that everything is reducible to logical functions we do not admit. The only excuse for such a notion might be in the tacit interpretation of "pure" mathematics so as to exclude any *proposition*, as for instance, *this* triangle is isosceles. But here the old question: What is an individual? is met.

Another notion introduced by logistic is that of truth and truth-value. In no place are either of these terms made clear, or are they defined. They are qualities of *propositions*, that is, propositional functions which have had individuals inserted for the variables. For example, if I consider the propositional function: x is right-angled, and then for x insert, respectively, the triangle ABC, the parallelogram S, this pink color, I have the assertions: ABC is right-angled, the parallelogram S is right-angled, this pink color is right-angled. The first of these is said to have the truth-value *truth;* the second, the truth-value *false;* the third has the value *absurd*, which is not a truth-value. The first two assertions are then propositions, the third is not a proposition. Much

is made of the idea of truth-value, but practically it amounts only to saying that an assertion is a proposition only when it can be labeled with one of two given labels. If any other label is necessary, it is not a proposition and not within the region of logic. So far as really used in logistic, these labels are neither more nor less than labels of consistency and inconsistency. They do not refer in any way to objective truth. Thus, if we start with the postulates of Euclidean geometry, we arrive at certain propositions, as, "triangle *ABC* has the sum of its angles equal to two right angles." This proposition is not to be tagged as true, but merely as consistent with the premises with which we started. The determination of the primitive truth of the premises is not possible by logistic at all. The whole of science is of this character, the truth of the conclusions of science being only probable, not certain, although the reasoning is valid. Science draws its validity from the agreement of all its conclusions with experience. In the same way the conclusions of mathematics are consistent under our notions of consistency, but neither true nor false on account of the reasoning. And this is all that Russell is privileged to say when he asserts that "mathematics is the science in which we do not know whether the things we talk about exist nor whether our conclusions are true." From the results of logistic we certainly do not know either of these things. We merely know that, if they exist and if the premises are true, then the conclusions are true, provided the processes of logistic can give true conclusions. Since logistic does not touch the natural world, and since everyone admits that mathematics does give us truth, the only possibility left to Russell was to assert the existence of a supra-sensible world, the world of universals

of Plato in another form. In mathematics, he says, we are studying this world and are making discoveries in it. It exists outside of the existence of any individual mind, and its laws are the laws of logistic naturally. That such world exists we will readily admit, but we deny that it stands finished as a Greek temple in all its cold and austere beauty, but that it is rather a living organism, a product of creative evolution, similar to the earth in geologic times, and out of the stress of temperature and moisture and dazzling sun there is evolved through the ages a succession of increasingly intricate and complex forms. But these forms derive their existence from the radiant energy of the human mind streaming into the chaos of the unknown. Even logistic itself is the outburst of the mind from the barriers of the early attempts to think and to think clearly. Mathematics finally attacked the process of thinking itself, just as it had considered number, space, operations, and hypernumber, and created for itself a more active logic. That this should happen was inevitable. Says Brunschvicg[1]

> Symbolic logic, like poetic art, following the spontaneous works of genius, simply celebrates the victory or records the defeat. Consequently it is upon the territory of positive science that the positive philosophy of mathematics should be placed. It gives up the chimerical ideal of founding mathematics upon the prolongation beyond the limits imposed by methodical verification itself of the apparatus of definitions, postulates, and demonstrations; it becomes immanent in science with the intention of discerning what is incorporated therein of intelligence and truth.

The philosophic assumption at the root of the view taken by the supporters of logistic as the sole source

[1] *Les étapes de la philosophie mathématique*, p. 426.

of truth we are not much concerned with, since we are not discussing philosophy but mathematics. But we may inspect it a little with profit. This assumption is the very old one: that there is an absolute truth independent of human existence and that by searching we may find it out. Says Jourdain[1]

> At last, then, we arrive at seeing that the nature of mathematics is independent of us personally and of the world outside, and we can feel that our own discoveries and views do not affect the truth itself, but only the extent to which we or others can see it. Some of us discover things in science, but we do not really create anything in science any more than Columbus created America. Common sense certainly leads us astray when we try to use it for purposes for which it is not particularly adapted, just as we may cut ourselves and not our beards if we try to shave with a carving knife; but it has the merit of finding no difficulty in agreeing with those philosophers who have succeeded in satisfying themselves of the truth and position of mathematics. Some philosophers have reached the startling conclusion that truth is made by men, and that mathematics is created by mathematicians, and that Columbus created America; but common sense, it is refreshing to think, is at any rate above being flattered by philosophical persuasion that it really occupies a place sometimes reserved for an even more sacred being.

Doubtless if Columbus were to discover America over again, he might conclude that acts of creation had gone on in the meantime, and might reasonably assume that they had happened in the past, and doubtless Mr. Jourdain is forced to conclude from his own argument that the words he uses in the English tongue have not been built up by the efforts of man, but have existed from the

[1] *Nature of Mathematics*, p. 88.

beginnings of time, that the idea of propositional function and of relative and of function, point-set, transfinite number, Lobatchevskian space, and a long list of other terms have always been waiting in the mines of thought for the lucky prospector, but common sense would refute this view with very little study of the case. We may grant that electric waves have always existed, but that the wireless telegraph has always existed in any sense is not true; nor that even if carbon, nitrogen, hydrogen, and oxygen have always existed, nitroglycerine is to be dug out of wells, or that, because sound-waves exist in the air, therefore symphonies, operas, and all music have always been waiting to be discovered, not created. It is true perhaps that the elementary units which compose things material or mental exist in some sense, external to any one individual in some sense, but it is not true that therefore the combinations of these elements have always existed. Logistic, with all its boasted power, has never constructed a theorem that was truly synthetic in character, it has never taken a set of new postulates not derived from previously existing theories and developed a branch of mathematics similar to geometry or algebra. It is powerless to move without the constant attendance of the intellect, it draws no more conclusions than Jevons's logical machine without its operator. It has never even introduced as one of its results a new thought of wide-reaching power, such as the idea of propositional function itself. This idea came from the extension of the mathematical function to other things than quantity. Columbus did not create the trees or Indians or shores of America, but he did create something that the Icelanders and Chinese or other reputed previous discoverers did not create, and its existence we celebrate today more than the

forgotten Indians, or the shifting sands of Watling's Island, or the broken tree trunks. Mathematics, as we said before, did not spring like Athena from the head of Zeus, nor is it the record of the intellectual microscope and scalpel, but rather as Pringsheim,[1] who is not a philosopher but a mathematician, says:

> The true mathematician is always a great deal of an artist, an architect, yes, of a poet. Beyond the real world, though perceptibly connected with it, mathematicians have created an ideal world which they attempt to develop into the most perfect of all worlds, and which is being explored in every direction. None has the faintest conception of this world except him who knows it; only presumptuous ignorance can assert that the mathematician moves in a narrow circle. The truth which he seeks is, to be sure, broadly considered, neither more nor less than consistency; but does not his mastership show, indeed, in this very limitation? To solve questions of this kind he passes unenviously over others.

We must pass on, however, to the reef that wrecked logistic in its short voyage after imperial dominion. This is nothing less than infinity itself. Since logistic asserted philosophically the supra-sensible and supra-mental existence of its objects, it was forced to assert that there is an absolute infinity. In the transfinites of Cantor it found ultimately its ruin. In order to handle collections that had an infinity of members it had to set up definitions that ultimately led to the contradictions which in the *Principles of Mathematics* of Russell were left unsolved. These were the objects of the assaults of Poincaré and others, and led to the definitive abandonment of the second volume of the *Principles*. The presentation of the *Principia* has many modifications, too long to cite,

[1] *Jahrb. Deut. Math. Ver.*, 13 (1904), p. 381.

but the discussions in the *Revue de Métaphysique et Morale* from 1900 on will be found very illuminating in their bearing on the nature of mathematics. The philosophical writings of Poincaré particularly should be consulted. The net result of all the discussions is that all the metaphysics has been eliminated from logistic, and it assumes its proper place in the mathematical family as a branch of mathematics on a par with the other branches we have considered or will consider, such as arithmetic, geometry, algebra, group-theory, being, in fact, closely allied to the subject of the preceding chapter, the theory of combination; it is indeed the theory of foundations.

The question of infinity is one of the most difficult to consider, and in one of his last articles Poincaré despairs of mathematicians ever agreeing upon it. The reason he gives for perpetual disagreement is the fundamental difference in point of view of reasoning in general. If the objects of mathematics are supra-mental, then the mind is forced to admit an absolute infinity. If the objects of mathematics are created by the mind, then we must deny the absolute infinity. So far no decisive criterion has appeared, beyond that laid down by Poincaré, that any object about which we talk or reason must be defined, that is, made to be distinguishable from all other objects, in a finite number of words. For example, there is no such thing as the collection of all integers, since, while we may define any one integer, we cannot define each and every integer. When logistic seeks to correlate the collection of all integers to any other infinite collection, member to member, this criterion demands that a law of correlation be stated which may be applied to every member of the collection. This is manifestly impossible. A case is the proof that rational numbers may be put into a

one-to-one correspondence with the integers. While any one rational may be placed in this way, or any finite number of them, yet, according to the criterion, it is not possible to decide that we can place *every* rational in this way. Manifestly any operation that has to be done in successive steps will never reach an absolute infinity. All proofs relating to infinite collections consider that the statement of a law for any one is sufficient. The criterion demands a law for every one, which is admittedly not possible. The absolute infinity must not be confused with the mathematical infinity, which is merely an unlimited or arbitrary class. In all the processes we use in getting limits, the infinity that enters is not the Cantor transfinity. Nor is an infinite class an infinite collection.

We may, then, safely conclude that logistic furnishes truth to the other branches of mathematics in exactly the same way that algebra does to geometry, or geometry to algebra, or numbers to group-theory, or hypernumbers to geometry. By logistic we may draw conclusions about the elements with which we deal. If we try to interpret the conclusions, logistic is powerless to do so any more than geometry can yield us theorems in logic. Also, the processes of reasoning of any nature are no different in logistic from what they are in algebra, geometry, theory of numbers, theory of groups, and it is the reasoning, not the logistic, that draws the conclusion of logistic, just as it is the mathematician that solves algebraic equations, not algebra. Logistic has a right therefore to exist as an independent branch of mathematics, but it is not the overlord of the mathematical world. As to the philosophical import of logistic, we may well follow Poincaré's advice and continue the development of mathematics with little concern whether realism or idealism or positivism

is substantiated in the philosophical world. Indeed, we may conclude eventually with Lord Kelvin[1] that: "mathematics is the only true metaphysics."

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[1] *Life*, p. 10.

CHAPTER VI

OPERATORS AND THE REDUCTION OF MATHEMATICS TO ALGORITHMS

The notion of change is very old. Since the first dawn of consciousness man must have watched the everlasting march of events in an irreversible procession. He has seen his hopes realized perhaps sooner or later, only to vanish into an irretrievable past. The sun rises and sets, and the stars pursue their solemn course over the sky. Summer follows spring, only to be followed by autumn and winter. The years roll past into centuries, and these become millenniums, and these ages. The newborn moment is already dead, and the present is a mere nothing between two infinities.

The school of Heracleitos twenty-four centuries ago made incessant "becoming" their universal principle of philosophy, and for the universal principle of the world of nature they selected the restless Fire. "Into the same stream we descend," he says, "and at the same time do not descend, for into the same stream we cannot possibly descend twice, since it is always scattering and collecting itself together again, forever flowing toward us and at the same time away from us." Hence we should learn to regard those elements of the eternal flow that persist and are unchangeable, he adds in his practical philosophy. In a later chapter we will consider mathematics from the standpoint that views it as the science which determines the invariants of human experience, but at present we are interested in it as the science that is concerned with

transmutations of form or substance. The explicit use of this very ancient notion did not occur in mathematical evolution until comparatively late. True, when the Greeks derived the conic sections as sections of one and the same surface, they might have thought of them as shadows of a circle made by a point of light and in this way have come to think of them as produced from a circle by a single operation, but they did not evolve this thought. The fact was as intuitively present in their knowledge of the world as any other mathematical fact, for they all had seen shadows of wheels and rings. But the school of Heracleitos was soon scattered, and his books called obscure. The Greek sought only the unchangeable, the absolute, the eternal. The idea of evolution was yet to be born to the world of thought, and as for creative evolution, it has only recently appeared in the new philosophy of everlasting change, the philosophy of Bergson. There are now all the more certainly eternal principles still to be found, but these are not preassigned, from the beginning to the end, rather are they unforeseeable from the past, and indeed, only when the occasion comes do they burst forth as new forms of thought. The arrow in its flight occupies a series of positions, constituting the trajectory that it had, and this dead thing may be studied by mechanics, and the successive positions correlated under a law. The law we call an explanation of the trajectory, and we hope by its means to predict trajectories again. Yet we really know not whether the arrow would ever retrace its precise path. Our laws are approximations drawn from a set of random points, and absolute precision is impossible for the moving thing. Science studies that which has been, knowing nothing of that which might have been.

But the dream of science to find absolute invariants in the everlasting flux we may realize better in mathematics, for we may study the life of what we ourselves create from a better vantage point than that of observation.

If Pythagoras' school had been more mathematical and less mystical, we might fancy them discovering facts more significant than that 10 was a mystic number. They might have observed that all the integers could be produced by the successive addition of 1, and this operation they could have symbolized by $+1$, attaining thus the notion of an iterated operator. They might have arrived at the notion of representing a variable number by a single letter rather than a fixed number, and so could have produced a formula for the addition of b 1's

$$X=x+1+1+1 \ldots +1,\ X=x+b.$$

If their inventiveness had progressed rapidly, they would have devised an inverse operation, called subtraction of 1 and indicated by -1, such that $(x+1)-1=x$. In such case they would have no doubt invented a symbol for zero, the number produced by the operation $1-1$. It would have had the curious property

$$0+x=x,\ x-0=x.$$

From this point of view they would have seen that there were negative numbers, which would make subtraction always possible.

Let us suppose that they would next attack multiplication, discovering the effects of operating by 2, by 3, 5, 7 ... and giving exactly what Eratosthenes found in his famous sieve. A single operator would not be sufficient to generate all integers, as it would in addition. Eratosthenes discovered this fact by striking out of the list of

integers every second, then every third, every fifth, etc., finding that there were always some numbers left over with which to begin new series. These new generators they could have called with propriety prime, that is, first numbers. Indeed, they would have seen that for multiplication it was necessary to have the generators, $-1, 1, 2, 3, 5, 7, \ldots$ With these they could have investigated multiplication, and by inventing the inverses, or unit fractions of the Egyptians, would have been able to divide in all cases. It might then have dawned upon the brightest minds that these four operations and the numbers invented for their action formed a closed set, the result of any combination of the operations being a number of the set. That is to say they could have set down the theorem that the totality of operations represented by

$$x' = ax + b,$$

where a and b, x and x', might be positive or negative, integral or fractional, always yielded a number of the set. This set of numbers and operations could have been called the rational set, being so very reasonable. The operations are infinite, that is, unlimited in number, and the succession of any two would not give numbers that could be called next to each other, since another operation could be found to give an intermediate result, hence they could have said that the rational operations form a discontinuous infinite group.

They might have been enough interested to have gone farther. They would have found, for instance, that if S indicates the operation on x given by $x' = a_1 x + b$, where a_1 is a given number, and b is in turn each of the rational numbers, the set of operations S would always give a number of such a character that if we construct

any other operation T, such as $m(\ )+n$, and understand by T^{-1} the operation $\frac{(\ )-n}{m}$, then the succession of operations, T, S, T^{-1}, will, whatever m and n (excluding $m=0$), always give a number in the set produced by the operations S. Indeed, starting with x, we would have in succession

$$x;\ \text{after } T,\ mx+n;\ \text{after } S,\ a_1mx+a_1n+b;\ \text{after } T^{-1},$$
$$a_1x+\frac{(a_1-1)n+b}{m}.$$

The last operation, considered as starting from x, is evidently of the type S. This operation they could have called conjugate to the operation S first used and could have said that it was conjugate under the transformation produced by T. The totality of operations S would then constitute a conjugate class. In case a_1 were changed to a_2 we should have a second conjugate class, and, indeed, there would be an infinity of conjugate classes, one for each number, a. The particular one for which $a=1$, $x'=x+b$, where b is in turn every number of the set of rationals, would easily be seen to be also a subgroup, the group of additions, and, since it is a subgroup conjugate to itself, it would have been called an invariant subgroup. It is easy to see that multiplications, $x'=mx$, do not remain invariant, although they form a subgroup, the operation of two successive multiplications being equivalent to a single multiplication. The conjugate, indeed, of $x'=ax$ would be $x'=ax+d(1-a)$, where T is written in the form $m(x-d)$. This could also be written in the form $x'-d=a(x-d)$.

A little reflection would have shown them that it would be possible to find a single operation which would convert

any two given numbers into any two others, arbitrarily assigned, since this would be equivalent to solving the two equations in a and b,

$$x_2 = ax_1 + b \quad \text{and} \quad y_2 = ay_1 + b,$$

which give

$$a = (x_2 - y_2)/(x_1 - y_1),\ b = (x_2y_1 - x_1y_2)/(y_1 - x_1).$$

This property they could have labeled double transitivity of the group of rational operations.

It might have been evident by reflection that, if a set of operations could be found that would leave some one number unchanged, they would form a subgroup. For example, multiplications leave 0 unchanged and evidently form a subgroup. If any such set is H, and H leaves x unchanged, and if T converts x into y, then the succession of operations T^{-1}, H, T, would evidently leave y unchanged; for the results would be: after T^{-1}, x; after H, x; after T, y. The set conjugate to H would thus leave the number y invariant. For instance, to multiplication, which leaves 0 invariant, is conjugate the set $x' - d = a(x - d)$, which evidently leaves d invariant. Such a set would form a subgroup conjugate to the group H.

We need not push the fancy any farther, the notion of group did not exist at that time, not even the notion of operator. All they had thought of was the set of numbers, integers, and fractions. However, it seems evident from the fancy that the notion of group and operator emanates from the mind's attempt to view its objects from its own standpoints, and not to analyze the objects for residues which may be called concepts, nor to introspect its own activities for such laws. If these notions are a priori, but only emerge in the course of time, then we

have substantially the same phenomenon as called for by creative activity. It was twenty-three centuries before these notions emerged into the consciousness of mathematicians.

Operations may be divided into two classes—the discontinuous and the continuous. In the first class it is not possible to find an operator that will produce from the operand A a continuous series of operands up to B; in the second class this is possible. We may sometimes find in the first class that we are able to select an operator that will produce from A an object as close to A as we desire, but the operator will depend upon A and would not for other operands produce an object arbitrarily close.

As examples we may consider the operation of turning a radius through an angle θ. If this angle is commensurable with 360°, the different radii will be finite in number, and eventually one of them will coincide with the first. The operation is called in this case finite, as well as discontinuous. If θ is not commensurable with 360°, then we can eventually by repetition of the operation produce a radius as near to the initial radius as we please, but the intermediate radii do not form a continuous set. We have a different case in the operation of adding 1 to 0, the successive numbers never approaching indefinitely close.

If the angle above is allowed to take any value, then the radii resulting form a continuous set, and the operations constitute a continuous group. The notion is recent, the possibility of it is, of course, as old as the race. It might have occurred to Euclid. Says Poincaré:[1]

In fine, the principal foundation of Euclid's demonstrations is really the existence of the group and its properties. Unquestionably he appeals to other axioms which it is more difficult

[1] *Monist*, 9 (1898), p. 34.

to refer to the notion of group. An axiom of this kind is that which some geometers employ when they define a straight line as the shortest distance between two points. But it is precisely this kind of axiom that Euclid enunciates. The others which are more directly associated with the idea of displacement and with the idea of groups are the very ones which he implicitly admits and which he does not even deem it necessary to state. This is tantamount to saying that the others were assimilated first by us and that consequently the notion of group existed prior to all others.

The group referred to here is called the group of Euclidean movements and consists of all translations and rotations in ordinary space. Its equations in finite form are the well-known equations for transforming to new rectangular axes. In other words, those properties of figures that do not depend upon a choice of axes are the ones with which Euclid is concerned.

However, we can scarcely agree that the notion of group had ever emerged into human consciousness and had become so familiar that it was ignored. The Greek, of course, observed that handling figures of material objects and moving them around did not disturb their size and shape, but the observation brought no intellectual response from him, any more than it did from a beaver who carried his logs of wood around, or from a bird who placed twigs in a nest. It took several centuries for the race to develop to the point where it could conceive experience as other than it is. Until this happened, the notion of group could not have existed. Such notions are simply not existent at all rather than stored up in an inherited mental storehouse.

We may loose our fancy again and suppose the school of Euclid of Alexandria discussing the question as to what

right they had to move their figures around and to superpose them on each other. They might have concluded that they could state that a translation of a figure in space, or of a rotation about a line, or of both combined in any manner did not disturb any of the properties of the figure with which they were concerned. They could have seen easily that the most general motion was a rotation about a line and a translation along that line. If such a motion is called S, and T is any other motion whatever, and if T converts s, the line of S, into a line t, then $TST^{-1}(t)=t$, for S does not change s. It follows readily that the necessary and sufficient condition that two motions be conjugate is that they have rotations of equal angle and translations of equal pitch. Evidently all translations for which the pitches are equal are conjugate, as well as all rotations for which the angles are equal. The successive operation of translations is a translation, hence translations form a subgroup, which is its own conjugate and is therefore invariant, represented by H. The group of motions, they could have seen, would have interchanged any two points, any two lines, any two planes; the group is thus simply transitive. The totality of motions that leave a point invariant are evidently the rotations about axes through that point, which thus form a subgroup. If we reduce every motion modulo the group H of translations (that is, consider for every motion that the pitch is zero), the group G so considered is homomorphic with a group indicated by G/H, called the quotient group, which is evidently the group of rotations, a subgroup here.

Motions which bring a body into congruence with itself form subgroups. As examples they could have found the tetrahedral group containing twelve operations, the octahedral group for the cube and the octahedron

containing twenty-four operations, and the icosahedral group containing sixty operations. They might even have found that the group that leaves a plane invariant is continuous in part and discontinuous in part, since translations parallel to the plane and rotations about axes perpendicular to it would leave it invariant and are continuous, but rotations about axes in the plane would have to have angles of $180°$ and would thus be discontinuous.

We see how much, then, the school of Euclid missed, not because they had no intuitive power, not because they could not reason, not because they could not draw physical figures, but because the notion of continuous group did not yet exist in the world of thought. It was twenty-one and a half centuries before Lie developed this notion, almost by his own unaided power. This is a most striking example that the great mathematician may receive his hints from nature, or from logic, or from an attempt to find common features and thus to generalize, yet it is his own living genius that brings forth the really living products of thought.

The group notion in any form did not emerge until near the close of the eighteenth century, appearing then in connection with the problem of solving algebraic equations. Such groups were made up of operators that permuted a given set of n elements among themselves, as, for example, the roots of an algebraic equation of order n. These developments occur in the work of Lagrange and Vandermonde, in 1770. The important series of developments of Galois and Cauchy that followed this algebraic beginning occupied so much of the attention of the mathematical world for a long time that, as Klein[1] says: "One considered in consequence of this

[1] *Höhere Geometrie*, 2, p. 4.

point of view groups as furnishing an appendix to a treatise on algebra, but certainly incorrectly. For the notion of groups appears widespread and in almost every mathematical discipline." We find a similar remark in Lie-Scheffers:[1] "In recent times the view becomes more and more prevalent that many branches of mathematics are nothing but the theory of invariants of special groups."

An example of continuous group is the totality of homogeneous linear substitutions on n variables

$$x_i' = \Sigma a_{ij} x_j \qquad i = 1, \ldots, n$$

a group which is of great importance geometrically. If we limit the coefficients a to be integers, we have a discontinuous group of much importance in the theory of numbers. The totality of projections in space of three dimensions gives us the projective group; and its invariants, projective geometry. The totality of conformal transformations of figures in a plane is given by the set of analytic functions of $z = x + iy$, which indicates the use of an analytic function when considered to be an operator on z.

The theory of continuous groups is due almost wholly to Lie, a Norseman, who studied the field of integration of differential equations very thoroughly, and thus came to produce his work upon transformation groups. He was studying this theory when he was surprised in Paris in 1870 by the Franco-Prussian War. Retiring to out-of-the-way places in Fontainebleau, so that he would not be interrupted, his diagrams aroused the suspicions of the police, and he was arrested as a spy. Darboux, however, hearing

[1] *Continuierliche Gruppen* (p. 665).

of it, soon convinced the authorities that his calculations would not assist the Germans to capture Paris, and he was promptly released. In the theory of continuous groups we see a notion that has been evolved not at all from a study of nature, but from the profound insight of the founder into the very intricate character of differential and integral formulae. The notion once emerged has spread its wings and has penetrated regions to which it was at first quite foreign.

We turn our attention now to the discontinuous groups. These may contain an infinity of operators or only a finite number of them. As a very simple example we may consider the group of all transformations of the form:

$$x'=\frac{\alpha x+\beta}{\gamma x+\delta} \qquad \alpha\delta-\beta\gamma=1,$$

where α, β, γ, δ are integers. Evidently α is prime to β and γ, as is δ likewise. As two such transformations we have, for example,

$$x'=\frac{2x+1}{5x+3} \qquad x''=\frac{4x+5}{3x+4},$$

their "products" being

$$(x'\cdot x'')=x'''=\frac{11x+14}{29x+37} \qquad (x''\cdot x')=\frac{33x+19}{26x+15}.$$

This class of substitutions is of great service in the study of equivalent quadratic forms in the theory of numbers. By their means we unify the theory and connect it with similar investigations elsewhere. A very useful related group is the modular group, in which all the numbers are taken modulo some prime, that is, every number is divided by the prime and only the remainder retained.

We find also the group of linear fractional substitutions of complex variables of great importance in the study of periodic functions and automorphic functions in general, as, for example, the elliptic functions and the polyhedral functions. These are related to the solutions of algebraic equations by the fact that the variables in an algebraic equation may be uniformized in terms of automorphic functions and thus lead to solutions. For example, we remember that the general quintic depends for its solution upon the expression of the variable x in terms of the solution of the icosahedral equation. We find the details in Klein's lectures on the icosahedron. We arrive thus at the finite discontinuous group, which was the starting-point of the theory of groups.

The brilliant work of Galois in the application of finite groups to the theory of equations is well known. Starting with Cauchy and continued by Jordan, whether in the form of substitution groups or in the form of linear groups, we find a very extensive development. A branch of mathematics of little more than a century's progress has thus been evolved from the study of purely mathematical notions. Yet it is a branch with which one must be acquainted in order to study any part of mathematics from arithmetic to the applications of mathematics to physics. Indeed, it is the notion of group that has upset physical theories and made it impossible to retain all the old notions. It is thus evident that the group does not owe its origin either to intuition or to physical law. It is to be found in the creative attempt of the mind to devise a means of solving an algebraic equation, to reduce one form to another, to transform a geometrical figure into another. And at the base of the notion of group is the fundamental notion of the operator, the transmutation of things that

are fixed into other forms, the symbol of change. The center of interest has been shifted from the fixed to the changing. We enter the stream, not to let it flow past us, but to be carried along with it. We have comprehended change.

We thus come back to the fundamental notion with which we are dealing—the operator that converts one object into another. The last example we had in the finite group as applied to equations owes its importance to the fact that all the roots of the icosahedral equation may be produced from any one of them by the operation of the linear substitutions of the icosahedral group. But the genius of Galois was necessary to bridge the gap from the work of Lagrange on symmetric functions and his resolvent equations. The notion of operator was thus the means in a double sense for solving the problem.

Operators in general owed their origin to other ideas than the solution of algebraic equations. The first to conceive of an operator as an entity was Leibniz, who perceived the similarity of the differential formula for a product to the expansion of a binomial. Lagrange took the definite step of separating the operator from the operand and gave the well-known formula of finite differences.

$$\Delta^n u = (e^{hd/dx} - 1)^n u.$$

Several of his successors gave formulae of a similar character, but with artificial and unsatisfying proofs. The first real appearance of the notion of operator on a substantial basis was in an article by Servois.[1] He showed that the properties of the operators under consideration were due to the formal laws of their combinations. His

[1] *Ann. math. pures et app.*, 5 (1814–15), p. 93.

work was carried still farther by Murphy[1] and Boole.[2] Upon their developments rest those of the present day. These laws are derived by considering the operators to act only upon a general range. They lead to (1) equality of operators; (2) uniformity and multiformity; (3) sum; (4) product; (5) multiplication, facients; (6) correlative multiplications; (7) limitation-types, as commutativity, or others; (8) simultaneous statements; (9) iteration; (10) distributivity.

The present-day developments are due to Pincherle,[3] Bourlet,[4] Moore,[5] Fréchet (papers on line-functions, etc.). The range in these later ones is a range of functions, and the operator converts a function into something else. So important and widespread has the notion become that one is tempted to assert that the whole of mathematics could be expressed as the result of purely arbitrary modes of combinations of operators, the applied mathematics consisting in the assignment of the range of the operators. This would be the reduction of mathematics to the science of algorithms. We would thus be led into a pure formalism which, while it might demonstrate the ability of the mind to build castles in Spain, would not give them the truth of reality which mathematics possesses. Though of vast importance, the theory of operators and of groups is only one of the many ways in which the mind attacks its problems. It is not the goal of mathematics, nor the means of solving all problems. The mind has faced the problem of

[1] *Phil. Trans. Roy. Soc. Lond.*, 127 (1837), p. 179.

[2] *Phil. Trans. Roy. Soc. Lond.*, 134 (1844), p. 225; and *Math. Analysis of Logic* (1847), pp. 15–19.

[3] *Encyc. des sci. math.*, Tome II, Vol. 5, Fasc. 1.

[4] *Ann. École Normale* (3), 14 (1897), p. 133.

[5] *General Analysis.*

change and transmutation and has responded with a vast creation, just as it faced the problem of the collection or ensemble, the problem of dimensionality, the problem of synthesis of elements, the problem of concept, relative, and others, in each case responding with a sublime creation. It does not, however, forget that there are others of its creations equally vast. And in the future we shall be able to create new ideas which, like number, space, combination, operation, will open vistas of new worlds that we have made possible. In tracing the origin and growth of these, as well as the others yet to be considered, we will become all the more assured as to the character of the future.

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CHAPTER VII

HYPERNUMBERS AND THE REDUCTION OF MATHEMATICS TO ALGEBRA

The problem we approach next is one that is very closely related to the general question of the validity of mathematics. It is of fundamental importance in the philosophy of mathematics. We have already seen how the notion of number was generalized from that of the simple integer to the fraction, the irrational, and the point-set. We have now the consideration of a generalization which is of a different character, and which gives us the branch of mathematics called algebra, just as the other gave us the general theory of numbers. In order to make a sharp distinction between the objects of study, we shall call the present objects hypernumbers. In the first division of the content of mathematics which we considered, and which may be called the static side of the subject, we started with numbers, including ensembles of various kinds as elements. The second division had as elements what we called manifolds, or multiplexes, or vectors. These were nothing more than elements which possessed more than one series of numbers, as, for example, in ordinary space the triplex vector is essentially, from the number point of view, three distinct series of values of elements which are themselves ordinary numbers. In the vector analysis of the plane we would, for example, be concerned with the duplexes (x, y). We may, it is true, represent the duplex (x, y) by a single symbol, say, z, and other duplexes in the same manner. We construct an algorithm, then, which we could call vector analysis

of two dimensions. Certain functions of the various couples of variables turn out to be of particular usefulness, as, for example, the duplexes (a, b) and (x, y) furnish the function $ax+by$, whose vanishing indicates that the vectors are perpendicular. The function $\sqrt{(a^2+b^2)}$ is called the length of the vector (a, b). Also we have the function $ay-bx$, which may be called the vector product of the two vectors and whose vanishing indicates that they are parallel. We may define a new duplex, which we call the product of the two duplexes (a, b) and (x, y), by the expression $(ax-by, ay+bx)$. Likewise in space of three or more dimensions we can do the same thing, and, indeed, this is what is usually done in the different treatments of vector analysis. We may look at complex numbers, such as $a+b\sqrt{-1}$ from this point of view, doing away with the imaginary. The inability of many mathematicians to take any other point of view than this leads to considerable confusion and lack of clear thinking, as is shown, for example, in some of the remarks about quaternions. Thus we find DeMorgan[1] saying: "I think the time will come when double algebra will be the beginner's tool; and quaternions will be where double algebra is now." On the other hand, we hear Lord Kelvin[2] complain: "Quaternions came from Hamilton after his really good work had been done; and, though beautifully ingenious, have been an unmixed evil to those who have touched them in any way, including Clerk Maxwell."

We may repeat the opinion of the philosopher Russell,[3] on the imaginary and on related objects:

All the fruitful uses of imaginaries, in Geometry, are those which begin and end with real quantities, and use imaginaries

[1] *Life of Hamilton*, Vol. 3, p. 493. [2] *Life of Lord Kelvin*, p. 1138.

[3] *Foundations of Geometry* (1897), p. 45.

only for the intermediate steps. . . . To speak, for example, of projective properties which refer to the circular points, is a mere *memoria technica* for purely algebraical properties; the circular points are not to be found in space, but only in the auxiliary quantities by which geometrical equations are transformed. That no contradictions arise from the geometrical interpretations of imaginaries is not wonderful; for they are interpreted solely by the rules of algebra, which we may admit as valid in their interpretation to imaginaries. The perception of space being wholly absent, Algebra rules supreme, and no inconsistency can arise.

The opposite view is to be found in the remarks of Benjamin Peirce:[1]

This symbol, $\sqrt{-1}$, is restricted to a precise signification as the representative of perpendicularity in quaternions, and this wonderful algebra of space is intimately dependent upon the special use of the symbol for its symmetry, elegance, and power. The immortal author of quaternions has shown that there are other significations which may attach to the symbol in other cases. But the strongest use of the symbol is to be found in its magical power of doubling the actual universe and placing by its side an ideal universe, its exact counterpart, with which it can be compared and contrasted, and, by means of curiously connecting fibers, form with it an organic whole, from which modern analysis has developed her surpassing geometry.

That the imaginary has been of tremendous importance in the history of the world no one will pretend to deny. The problem is to account for it and to explain how it yields truth. If, as Russell believes, the use of the imaginary points at infinity leads to real and valid results for actual space, then either all the geometrical analysis depends solely upon algebraic notions to carry

[1] *Amer. Jour. Math.*, 4 (1881), p. 216.

it through, in which case an imaginary point is as real as any other point, for in neither case are we really talking about points in space—or else part of the time we are talking about geometric entities, part of the time about something else, in which case it is hard to see how non-geometric things can prove anything about geometric entities. Yet no geometer has any scruples at any moment about using imaginary points just as freely as real points. They may even be introduced without any reference to algebra.

We are concerned in algebra with negative numbers, with the imaginary and the complex numbers, and with others, all of them coming under the one name hyper-numbers. The numerical element is not of particular interest, only the so-called unit or qualitative part of the number. To each unit there corresponds a range which is the arithmetical character of the hypernumber. These ranges may be finite or infinite and need not be continuous. Indeed, they may become themselves hyper-numbers. Thus to the roots of the equation x^2+2x+7, $-1\pm i\sqrt{6}$, correspond the numerical values 1 and $\sqrt{6}$ for the units -1 and i; but we write the same number in other forms as $(1\pm\sqrt{2})\omega+(1\mp\sqrt{2})\omega^2$, in which the units are now ω and ω^2 where $\omega^3=1$, and the numerical coefficients of the units are $1\pm\sqrt{2}$ and $1\mp\sqrt{2}$. We may also see incidentally here that these numbers are not the same duplexes in the two methods of writing, a significant fact. Indeed, we can choose a unit such that the root of the equation ceases to be a duplex at all, namely, if we write it as

$$(\sqrt{7})_\theta=\sqrt{7}(\cos\,\theta+i\,\sin\,\theta),$$

where

$$\cos\theta=-\tfrac{1}{7}\sqrt{7}\qquad\sin\theta=\sqrt{\tfrac{6}{7}}.$$

The fact that we use $\sqrt{7}$ to give the numerical part of the hypernumber and indicate the unit part by an index θ is of no more significance than to indicate a point by x_8. The unit is one thing and the index by which we identify it another.

We may suspect now that if we can account for algebra we shall have little trouble with the rest of mathematics. We meet the difficulty squarely face to face when we undertake to connect a negative number, say, -1, or the imaginary number $\sqrt{-1}$, with ordinary numbers. We must, however, really solve the problem and not merely evade it. That is to say, we must really deal with -1 and $\sqrt{-1}$, and not with something else we substitute for these because it may happen to be isomorphic with one or the other of these in some situations. Many apparent explanations have been given which have had this character. For instance, we find the definition given by Padoa[1] for a negative number. He writes for -1, *sym.* 1, meaning that -1 occupies in a linear scale a position symmetric to 1. It is, of course, obvious that if we reflect a linear scale of ordinary numbers in a mirror we will see in the mirror a scale that we may call a negative scale. These might be called the symmetrics of the original set. Again Peano[2] defines the imaginary to be the substitution:

$$\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$$

whence

$$i^2 = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} \qquad i^3 = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \qquad i^4 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}.$$

[1] *Bibliothèque du congrès internationale de philosophie*, 3, p. 325.

[2] *Formulario matematico*, 5th ed., Vol. 1, p. 152.

The complex number $a+bi$ becomes, then, the linear substitution

$$\begin{vmatrix} a & -b \\ b & a \end{vmatrix}.$$

But in the former case how is one to justify the statement $1+-1=0=1+sym.\ 1$, or in the latter the statement $i^2=-1$? Evidently the only way out is to assert that in the first case 1 and 0, as well as -1, are all only places on a scale. In the latter case 1, 0, -1, and i are all equally linear substitutions. Now, we are pretty well assured mathematically that i can be used like any other number. For instance, a theorem due to Gauss asserts that a prime of the form $4n+1$ is always factorable in the field of complex numbers into the product of two conjugate imaginaries, which is equivalent to a theorem of Fermat's that a prime of this form is always the sum of two squares. For instance,

$$5=(1+2i)\,(1-2i), \qquad 13=(3+2i)\,(3-2i),$$
$$17=(1+4i)\,(1-4i), \qquad 29=(5+2i)\,(5-2i)\ \ldots.$$

Did Gauss imagine for a moment that he was not dealing with numbers, but was solving problems about linear substitutions? Or did he imagine that his numbers were not 5, 13, 17, 29 .... but were the duplexes (5, 0), (13, 0), (17, 0), (29, 0) .... factorable as duplexes, but not as numbers? Because we have found an isomorphism, may we assert that we have found the thing itself?

The same question has really arisen before, in the definition of ratio. Is the list of ratios whose numerators are divisible by their denominators the same as the list of integers or merely isomorphic to them; and being a subgroup of a larger class, it becomes convenient to throw

overboard the integers and use the ratios of this type instead of them? When the negative was first introduced to solve such an equation as $x+5=3$, we may be quite sure the mathematician who invented the negative did not have in mind the relation *sym*. Nor did he think of 3, 5, and x as relations. Nor did the mathematician who created the imaginary in solving such an equation as $x^2=-4$ imagine for a moment that his x, and his -4 were linear substitutions and not numbers. Any such hypothesis manifestly is historically not true. Any other hypothesis than that these numbers were direct creations of notions necessary to complete the list of numbers is not tenable, and we may infer that the mind likewise must have created the integers in its endeavors to handle the world of objects. If some numbers are direct creations and not residues of phenomena, then all are.

The problem can be illustrated perhaps by a fresh example from the Kummer theory of ideal numbers. If we examine the number 6, we find that it has the factors 2 and 3, which are integers. But also if $\theta^2+5=0$, we find that we may express 6 as the product of the factors $1+\theta$ and $1-\theta$. There are, then, two ways to factor 6 into what we call integral factors. Now, the numbers 2, 3, $1+\theta$, $1-\theta$, cannot be broken up into integral factors in the form of either positive or negative numbers nor numbers of the form $x+\theta y$. The question arises, then, as to how it is that 6 can be factored into two different forms, which have no common factors themselves. If we consider that 24 can be factored into 8 and 3 or into 6 and 4, we see at once that the reason lies in the fact that 8 and 6 have the common factor 2, and that in 8 times 3 we have transferred the factor 2 from the 6 to the other factor 4, giving 8 times 3. But in the case above there is no such

possibility, since each one of each pair of factors is irreducible in the field. Kummer suggested, since the number 2 from other considerations behaves like a square in this domain, that we write it as the square of an ideal number, α, $2=\alpha^2$. For if we consider $x+\theta y$, x and y both odd, $(x+\theta y)^2=(x^2-5y^2)+2xy\theta$ is even and divisible by 2, while $x+\theta y$ is not divisible by 2. We find also, since $6=(1+\theta)(1-\theta)$ is divisible by 3, but neither $1+\theta$ nor $1-\theta$ is divisible by 3, that 3 behaves like a number with two ideal factors β_1 β_2. Thus we see that

$$6=\alpha^2\beta_1\beta_2=\alpha\beta_1\alpha\beta_2.$$

Now, it is obvious that we can factor 6, not only in one way, but also in another, and the other factors of 6 should turn out to be

$$1+\theta=\alpha\beta_1 \text{ and } 1-\theta=\alpha\beta_2,$$

which is the case. By the introduction of these ideal numbers, which are not in the domain of the integers and the compounds of θ, we have restored simplicity to the system. We are able now to state that every number of the domain $x+\theta y$ not equal to zero or ± 1 is either a prime number, that is, irreducible, or else it is a number which is a product uniquely determinable of numbers of the domain that are prime, or else of such numbers and ideal numbers. Thus 6 is the product of four factors, all ideal, which may be grouped in various ways. All these ideals may be expressed in terms of α and numbers which are ideal and of the form $(x+\theta y)/\alpha$. So that by putting α into the system we have preserved the laws of the system of integers, and at the same time we have extended our system of integers. The point we need to dwell upon here is that by the creation of new integers we have extended

the original list of integers. The fact that later we may write $\sqrt{2}$ for α and treat such an expression as $\beta_1=(1+\sqrt{-5})/\sqrt{2}$ as an integer, is merely another way of writing down the facts. These more elaborate forms do not disturb the essential character of the number. For example, we have not modified the character of the fraction $\frac{3}{4}$ by writing it with a double index, 3 and 4, and we should not let the mode of writing lead us off on false explanations, forgetting after two thousand years that the two numbers written down merely serve to identify the one fraction and that the fraction itself is only one number and is as much entitled to a single symbol as π or e, or just as 100 is a single number written with three characters. When fractions were created they were a distinct addition to the then-existing list of numbers, and we may say exactly that, in itself, and not looked at as an operator, the number 6 is not simply an isomorph to $\frac{6}{1}$, but is the same as the latter. A distinction between the two is due to some further idea than that of number. Likewise the irrationals, however indicated, were distinct extensions of the domain of numbers previously existing.

We are now in a position to see that the extension by means of the negative is also truly an extension and includes the previous system, which as a subclass may, it is true, be called arithmetic numbers, but which do not lose their identity or their existence when they are viewed as belonging to the larger class, in which relation they are called positive numbers. To go to the extreme of saying that a thing in and by itself and the thing as related to something else are not identical leads to absolute sterility in reasoning and to chaos in ideas. It justifies the claim of a Chinese wag, that a cow and a horse make three,

because there is the cow and the horse and the team, which make three distinct things. It is only the introduction of postulational methods which undertake to devise symbols to which are assigned certain explicit properties, which makes us forget the origin of these numbers. More exactly stated, these postulational entities are isomorphic with the realities that mathematicians deal with, and get their existence theorems from this isomorphism and not the reverse.

The negative number was devised to permit the solution of such equations as $x+6=4$. The creator of these fictive numbers (as they were called) had no other notion than that he was dealing with the same 6 and the same 4 that he had always been dealing with. That the use of these negatives was evaded as long as possible in the history of mathematics is merely a result of the law of mental inertia which shuns complexity and travels on the simple path as long as possible. But when the force of a love of harmony and completeness has reached a strong enough deflecting power, the straight line path is no longer possible, the mind creates a wider domain for its motion, and a new branch of learning is available for discoveries.

To whom the idea of the negative is due is not certain. It is ascribed to Diophantus by some, by others to the Hindoo mathematician, Brahmagupta, about 500 A.D., but it is certain that it appears in the algebra of Bhaskara (1150). In the fifteenth century Chuquet interpreted negative numbers and Stifel early in the sixteenth century speaks of absurd numbers, less than zero. Stevin late in the sixteenth century made use of the negative roots of equations, and in the seventeenth century Girard placed negative numbers on a par with the ordinary

numbers. It is true that a hundred years earlier Cardan had stated negative numbers as roots of equations, but he considered them as impossible solutions, mere symbols, a view held by some of his successors of the nineteenth century. Even Pascal regarded those who believed that they could subtract 4 from 0, as horrible examples of blindness in the face of shining verity.

The difficulty, of course, is very apparent now. It lay in the identification of integers with objects. Of course no one can take away 4 objects when there are no objects present at all. But taking away 4 objects and subtracting 4 from 0 are not at all the same thing. One might as well deny the existence of fractions, such as $\frac{1}{2}$, because, if one were to cut a man into 2 halves, he would not have $\frac{1}{2}$ a man, but only $\frac{1}{2}$ a carcass. There are some objects that do not admit the idea of fraction. There are some that do not admit the idea of irrational. There are some that do not admit the idea of negative. But there are others in each case that do admit the idea. While living bodies cannot be cut in half, as a rule, we may divide up a pile of sand into many fractions. While grains of sand do not admit of irrationals, yet the diagonals of rectangles do admit irrationals. While silver dollars do not admit being negative, one's bank account may, if properly secured. These various examples merely furnish occasions for the employment of negative numbers, irrational numbers, and the like, they do not prove their existence. The existence theorem of such numbers—indeed, of all numbers, including integers—is not to be found in concrete experience, but in ideal experience. Whatever is consistent with that structure of knowledge which the race has built up is ideally existent. The mathematician is rejoiced when his ideal constructions

are found to be practically useful, but he does not design them primarily for that end.

We may consider next the so-called imaginary numbers. The first explicit use of them appears to be in the solution of a cubic given by Bombelli, toward the close of the sixteenth century. Cardan had already furnished the formula which in the case of three real roots demands the extraction of a cube root of two complex numbers. In the problem cited, the equation is

$$x^3 = 15x + 4.$$

Cardan's formula leads to finding the cube root of $2+11\sqrt{-1}$ which is $2+\sqrt{-1}$, and of $2-11\sqrt{-1}$ which is $2-\sqrt{-1}$. The corresponding root of the equation is 4, the other two roots being $-2\pm\sqrt{3}$. A whole century went by before much was accomplished with the imaginaries. Then de Moivre in the eighteenth century (1738) gave his celebrated formula. Thirty-six years previously Leibniz and Bernoulli had seen that the decomposition of rational fractions for integration might lead to complex denominators, which produced logarithms of complex numbers. In 1714 Cotes showed that

$$\log. (\cos x + i \sin x) = ix.$$

During the eighteenth century the progress was fairly rapid, and toward the close of this century the complex number had become so generally recognized as a member of the mathematical family that various mathematicians undertook to find a geometrical justification for its existence. A Danish mathematician, Wessel, near the close of the eighteenth century developed the method now very commonly used for the representation of complex numbers in a plane. We under-

stand, of course, that this method of representation is simply one of many, and does not in any way make the numbers more or less real than they are without any geometrical representation. $\sqrt{-1}$ is not a unit line perpendicular to a given axis of reals. If we were to state exactly the significance of the representation, we should say that, if we consider vectors in a plane, then each may be considered to be the vector produced by attaching the idea of some complex number to a given vector considered as the unit or starting-point of the system. For instance, we do not multiply two vectors together, but we do multiply together the numbers that produce these vectors from the unit vector. The real situation becomes very easily seen in the Steinmetz representation of alternating currents and electromotive forces. The natural outcome of part of the geometrical method of representation was that the vector came to be drawn from a fixed origin, and the end-point of the vector was taken to represent the vector, then was taken to represent the number. Consequently, the complex number came to be looked upon as a duplex of two reals. That is to say, for the number which rotated the unit through the angle θ and stretched its length r times, came to be substituted the duplex $(r \cos \theta, r \sin \theta)$. The verity is, however, plain that the complex number is one entity and the duplex (x, y) another, and the duplex (r, θ), which represents the complex number equally well, still another. The derivation of the rules of addition of complex numbers from the corresponding duplexes, of course leads still farther away from the real facts and more into the pure artificiality we must endeavor to avoid.

From the beginning of the nineteenth century the theory of functions of a complex variable began to be

cultivated, and the place of the imaginary was now assured. For the detailed history, reference must be made to the *Encyclopédie des sciences mathématiques*, Tome I, Vol. 1, Fasc. 3. In 1833 Sir W. R. Hamilton undertook the treatment of the complex number as a couple or a duplex of two real numbers. His intention was to found algebra entirely on the notion of time, that is to say, succession, or as we might better say now, upon the notion of a well-ordered set. The chief importance of his investigation, aside from its being an early attempt at the arithmetization of mathematics, lies in the fact that he not only considered couples, but investigated triples and sets in general. This was the beginning of general algebra and was followed by the creation of the next extension of numbers, namely, quaternions.

Hamilton's creation of the quaternion numbers was due to the suggestion from the geometrical representation of the imaginary in a plane that there ought also to be numbers that would be represented by the vectors in space of three dimensions. This is a very good example of the interplay of mathematical methods. An analytical problem is put into a geometric form, and this in turn suggests some new analytic development that otherwise would not have been thought of. In the same way geometric problems, when put into analytic form, often have suggested by the analytic form obvious correlated problems or extensions. The object of Hamilton in this case was not to devise a geometric calculus, as has been sometimes incorrectly stated, but to extend the realm of numbers. His previous work on general algebra proves this, but fortunately we have his own account of how he came to invent quaternions,[1] which makes this also perfectly

[1] *Phil. Mag.* (2), 25 (1844), pp. 490–495.

clear. In quaternions there is a double infinity of imaginaries, any one of which, with real numbers, constitutes, the ordinary complex domain. They are linearly expressible in terms of any three of them which are linearly independent. The product of any two of the three imaginaries, however, introduced the first variation from the laws of combination previously existing, namely, that we have no longer commutative products, but skew, that is, $ij=-ji$, $jk=-kj$, $ki=-ik$, and two quaternions do not even have the skew multiplication of the three imaginary units, for, if the quaternions are:

$$q=w+ix+jy+kz$$

and

$$r=a+ib+jc+kd,$$

then

$$qr=rq+2i(yd-zc)+2j(zb-dx)+2k(xc-yb).$$

Hamilton spent the latter part of his life in developing the algebra of quaternions and its application to geometry of three dimensions. His work was carried on also by Tait and Joly. The functional side of the development of quaternions is slow, owing to the non-commutative character of the product.

Weierstrass investigated later the question of finding such extensions of ordinary algebra as retain the associative and commutative laws. His result is that there are no such extensions, beyond the use of various imaginary units like $\sqrt{-1}$, but whose products with each other vanish. This is a case of the creation of nilfactorial numbers. In such an algebra it is not possible to have unique division in every case. Indeed, the study of the domains in which division is unique for the continuous range shows that the only cases are arithmetic, ordinary algebra, complex numbers, and real quaternions.

We find the source of an endless series of discussions and disputes in the assertions that the product of two vectors may be a scalar or may not be, may be a vector or may not be, may be a bivector or something else. Of course the confusion comes from statements exactly similar to saying that 2 feet long times 3 feet wide gives 6 square feet. The real multiplication is that of the numbers 2 and 3. The feet are incidental. So the product of two vectors in space is an impossible thing. The only product is that of the numbers which the vectors represent. What such product is can never be determined from the vectors, but only from the numbers and their character. It has been urged that Hamilton was working only for a geometrical calculus. But besides the previous answer to this false conclusion we can point to his articles on the icosian game, in which he calls the numbers with which he is dealing, new roots of unity. They follow a more complicated law than that of i, j, k, and give, indeed, in their multiplication the icosahedral group. Later Cayley considered the operators of an abstract group as roots of a symbolical equation, thus making them hypernumbers.

We may sum up the result of the historical study of the development of hypernumbers as follows: The widest domain we have discussed yet is quaternions. If the hypernumbers are a subclass of quaternions of the form $x+iy$, then we have complex numbers. The subclass of this which consists of the so-called reals is the number domain of ordinary algebra. The positive subclass of this, again, is the domain of arithmetic.

This brings us to the consideration of what is the most general field of hypernumber, and we find that the question is unanswerable, just as there is no most general space

which contains all other varieties; but we may go on creating new number domains and new spaces without limit. Quaternions form a subclass of what has been called linear associative algebra, in whose domains of hypernumber the imaginaries, or better, the unit hypernumbers become numerous, indeed, even infinitely many. We find nilfactorial hypernumbers, nilpotent hypernumbers, and other varieties. The notion of hypernumber has widened out to cover a universe which is equal in extent at least to the universe of all kinds of spaces. The reduction of all mathematics to statements in terms of hypernumbers is called the reduction of mathematics to algebra. It has been said that mathematics tends always to the form of algebra and there is much truth in the statement, but it is not possible to reduce the entire field to any such comparatively simple result.

The first definite study of the general hypernumbers was made by Benjamin Peirce, in a lithographed memoir[1] in 1870. In this he introduced the notions of character, direct and skew units, although he did not use these terms. For the best account of the different memoirs on the subject see the *Encyclopédie des Sciences Mathématiques*, Tome I, Vol. I, Fasc. 3.

We need only to mention the most important of the extensions beyond quaternions. The first of these is without doubt the so-called Clifford algebras. In these we have a set of generating units from which we start, $i, j, k, \ldots\ldots$ such that the square of each is -1, and the products are skew, $ij=-ji, \ldots\ldots$ each product being a new unit, and all the products are associative. The unit of highest order is $ijkl\ldots.$ Including the ordinary

[1] Reprinted in the *American Journal of Mathematics*, 4 (1881), pp. 97–215.

unity we have for n generators, a system of 2^n different units in terms of which the algebra can be expressed. The products of even order are evidently commutative with any other products. Those of odd order are skew with each other. These algebras are applicable to space of n dimensions and include the Grassmann products of vectors as particular products.

The algebra of n^2 units of the form e_{ij} such that

$$\begin{aligned} e_{ij}e_{lm} &= 0 \qquad j \neq l \\ &= e_{im} \qquad j = l \end{aligned}$$

furnishes numbers which have all the laws of combination of matrices of order n. An algebra of this character is called a quadrate.

Whatever the form in which these various particular algebras may be studied, the hypernumbers are the abstract structures whose concrete forms are under consideration. Just as we may study groups in the form of substitution groups or linear groups, or other forms, while the basis remains the theory of abstract operators, so here we should not let any concrete application of the algebra obscure the fact that the fundamental theory is that of hypernumbers, which are the underlying abstract entities with which we really have to do. The notion of hypernumber, or algebraic imaginary, is distinct from that of numerical value or range; distinct from that of manifold; also distinct from that of operator. From this point of view the algebraic domains (*corpora*, fields) are included as particular cases of the hypernumbers of a simple type. For instance, the Galois theory of equations consists in the discovery of the particular algebra which contains the numbers necessary to solve the equation. For example, the roots of $x^4-22x^2-48x-23=0$ are rationally express-

ible in terms of the roots of $x^4-10x^2+1=0$. For the roots of the latter are

$$\sqrt{3}+\sqrt{2}=\rho,\ \sqrt{3}-\sqrt{2}=1/\rho,$$
$$-\sqrt{3}-\sqrt{2}=-\rho,\ -\sqrt{3}+\sqrt{2}=-1/\rho,$$

while the roots of the former are

$$\sqrt{2}+\sqrt{3}+\sqrt{6}=\tfrac{1}{2}(\rho^2+2\rho-5),$$
$$\sqrt{2}-\sqrt{3}-\sqrt{6}=(-\rho^3+5\rho-2)/2\rho,$$
$$-\sqrt{2}+\sqrt{3}-\sqrt{6}=(-\rho^3+5\rho+2)/2\rho,$$
$$-\sqrt{2}-\sqrt{3}+\sqrt{6}=\tfrac{1}{2}(\rho^2-2\rho-5).$$

The theory of groups shows that these equations have roots rationally expressible in terms of ρ. Another simple case consists of the abstract field of Moore.

The equations of which the hypernumbers are the roots are called the characteristic equations of the numbers. Their coefficients are in no way different from any other ordinary numbers. In hypernumbers we have, then, a most decisive example of the creative character of mathematics. These numbers and therefore all numbers parallel the natural world, as we see in the use of hypernumbers to study vectors, but they transcend it in the same way that hyperspaces and non-Euclidean spaces transcend it. The reality of these conceptions, however, is exactly the same as that of the ones which fit the natural world. Hypernumbers and hyperspaces have finally solved the riddle of the universe of mathematics and have even illuminated the shadows of the riddle of the world of natural science.

We find still further extensions of hypernumber in the functional transformations of integral equations. The transformation

$$\int_0^1 \kappa(s, t)(\quad)dt$$

may, for instance, be expressed as the sum of transformations like

$$\phi_1(s)\int_0^1 \psi_1(t)(\quad)dt + a\phi_2(s)\int_0^1 \psi_1(t)(\quad)dt$$
$$+ b\phi_3(s)\int_0^1 \psi_2(t)(\quad)dt + \ldots .$$

which are nothing more than idempotents and direct nilpotents. Other extensions are also in sight.

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CHAPTER VIII

PROCESSES AND THE REDUCTION OF MATHEMATICS TO TRANSMUTATIONS

The third large division of dynamical mathematics is the theory of processes, corresponding to the division of static mathematics called arrangements. By process is meant a mode of transition from a given object or arrangement of objects to another object or arrangement of objects. The wrinkled brow of the chess-player, as he studies the moves by which he can pass from the situation at hand to the checkmate of his adversary in twenty-four moves, indicates that even in such an amusement there may be work. The student endeavoring to turn his integral into a manageable, summable series is attacking a process. The table of function values of an elliptic integral enables the computer to pass from the argument to the value. The analysis of a given function from some definition given to some other form constructed out of elementary processes is one of the important problems of mathematics. Whether in the game, or in the puzzle, or in the elaborate mathematical theory, we have in this process of transition a mathematical character that would at first seem sufficient to define all mathematics. Processes occur everywhere in mathematics, and similar processes may appear in very different problems. Great mathematicians have not disdained the game as a means of reaching useful results. Euler studied the knight's move on the chessboard, the problem being to touch every square once. Hamilton invented the icosian

game as a pleasant way to visualize his new hypernumbers, of which the squares of some, the cubes of others, and the fifth powers of others were unity. The icosian game is to pass from one of the vertices of a dodecahedron to each of the others along the edges, so as to reach each vertex once. That mathematics may start in some amusement, such as this, is no detraction. Said Montesquieu:[1] "As there is an infinity of wise things conducted in a very foolish way, so there are follies conducted in a very wise way." In the game it is the combination, the transition, that pleases the mathematician, not the mere winning. In the play he may receive suggestions from which important theories flow.

We may easily reduce a large part of arithmetic to transitions. Starting with an object called the unit, we may consider that numbers are the modes of transition to other objects. For instance, by the process of addition we arrive at the whole integral scale, by the process of subtraction we arrive at the negative scale, by multiplication and division we reach the rational scale. By other processes we introduce the irrationals. We would then define the integers, the rationals, the irrationals, as stages of the iteration or combination of processes. We may generalize these processes more and more, reaching, finally, unspecified ordinal sets. Indeed, it has been shown many times that with certain generating relations it is possible to construct the whole theory of number. These generating relations are simple cases of processes.

In the same way by the processes of projection, of intersection, and others we need not stop to list, all projective geometry may be constructed. Other processes will give us metrical geometry. Modes of arrangement,

[1] Quoted by Lucas, *Récréations mathématiques*, title-page, Vol. 1.

the calculus of classes and relations, the theory of operations, algebra, may all be reduced to statements in terms of transitions or processes. So it might seem that we have at last a sufficient characterization of mathematics. This would seem to be more certain when we recognize that transitions have their invariants, their functionalities, and are a fruitful source of inversions by creation of new elements. Royce[1] undertakes to find herein a system, which he calls a "Theory of Order," that is fundamental for the philosophy of the future, and is to include all order-systems upon which the present theoretical sciences depend for the deductions. He gives it the following properties:

1. That the numbers, elements, or "modes of action" which constitute this logically necessary system Σ exist in sets both finite and infinite in number, and both in "dense" series, in "continuous" series, and, in fact, in all possible serial types.

2. That such systems as the whole-number series, the series of fractional numbers, the real numbers, etc., consequently enter into the constitution of this system. The arithmetical continuum, for instance, is a part of the system Σ.

3. That this system also includes in its complexities all the types of order which appear to be required by the geometrical theories recognized at present, projective and metrical.

4. That the relations amongst the logical entities in question, namely, the *modes of action*, of which this system Σ is composed, are not only dyadic, but in many cases polyadic in the most various way.

From the fact that modes of action, while they result in creations and ideal constructions, yet have certain necessary features, he seems to think he has herein found the long-sought absolute truth. The creations, he observes, however, are merely apparent, for the so-called

[1] *Encyclopedia of the Philosophical Sciences*, Vol. 1, p. 134.

"creation" of the order-types is "in fact a finding of the forms that *characterize all orderly activity, just in so far as it is orderly* and is therefore no capricious creation of his private and personal whim or desire." The only remark that we would add to this is that, while mathematics, it is true, registers the result of the activity of various mathematicians' minds collectively and thus shows the manner in which intelligent mind has acted, it is not therefore to be concluded because these are orderly and consistent that these modes of activity are not spontaneous acts and unforeseeable as to the future, which characteristic makes them creative. The scientist also registers the way matter has acted, but that he has exhausted its actions for all time he is not presumptuous enough to suppose. Mind seems to carry with it constantly its previous results, and once, having created, its creation never disappears.

However, we can scarcely say that we may now reduce all mathematics to mere problems in processes. The essential feature of the theory of numbers is not the mode of arriving at the continuum or other ensemble, but the ensemble itself. We might consider even that we have an ensemble of stages in a process, but so far as these are crystallized into individuals they cease to relate to transitions. The thing studied is the collection of individuals. In geometry the figure and its properties are the interesting thing, and not the mode of generation. In arrangements, in logistic, in theory of operations, in hypernumbers, the entities themselves are under discussion and these can scarcely be left out of mathematics. The mere parallelism or isomorphism between the theory of processes and the other theories testifies to the substantial unity of mathematics, but does not allow us to

abandon any one as superfluous. The abstract theory of groups, for instance, contains much that is true of substitution groups, but we cannot allow one to displace the other.

Processes have their invariants, for instance, the bishop in chess remains on his color; they have their functions, for instance, the move of one adversary is a function of the move of the other; they have their inversions, for instance, the chess problem. We also have equivalent groups of action, isomorphisms of action. There are ideal modes of action proposed to solve problems, and these are often the causes of human progress. The whole theory, however, is not yet very far advanced, and it will be some time before even existing mathematics will be stated wholly in terms of processes.

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CHAPTER IX

INFERENCES AND THE REDUCTION OF MATHEMATICS TO SYSTEMS OF LOGIC

The last great division of dynamic mathematics is that called theory of inferences. The speculative philosopher who is endeavoring to condense all mathematics into a single field and to frame a single definition for it might hope for a moment that, in the theory of systems of inference or systems of reasoning in every manner possible, he might find a solution to his problem. Even though the static results of the theory of classes and relatives do not contain all mathematics, yet in the active exercise of the thinking function of mind there might yet be a chance to enmesh the elusive spirit of mathesis.

The theory of inference would include all methods of drawing conclusions. Whether these depend upon the law of contradiction or upon more general laws in which there is, so to speak, a first contradictory, a second contradictory, etc., or whether they follow some generalization of the syllogism, or whatever the principles may be or the way in which the succession of data and consequences are related—all these come in the generalized theory of inference. Methods of analysis and synthesis of data, the symbolization of data, the transformations permitted, the elimination of extraneous elements, the statement of conclusions—these are within the field of structure of reasoning in general. Beyond these there are to be considered the invariants in modifications of the processes or the methods of deduction. These may be objectified into

laws of thought, or into laws of mind, or may be called laws of the universe, but whatever source they are ascribed to, their investigation belongs to the field of active reasoning. There will further be functionalities in reasoning, which are the systems that depend upon other systems and may be called functions of the latter. Finally, the solutions of general problems in reasoning, whether in the invention or in the discovery of systems of reasoning that will do a particular kind of work, or in the creation of modes of reasoning that so far the race has never evolved, or in the erection of scientific theories, will constitute the inversions of the systems of reasoning.

Said Peirce:[1]

> Mathematics is not the discoverer of laws, for it is not induction; neither is it the framer of theories, for it is not hypothesis; but it is the judge over both, and it is the arbiter to which each must refer its claims; and neither law can rule nor theory explain without the sanction of Mathematics. . . . . Even the rules of logic, by which it is rigidly bound, could not be deduced without its aid. . . . . In its pure and simple form the syllogism cannot be directly compared with all experience, or it would not have required an Aristotle to discover it. It must be transmuted into all the possible shapes in which reasoning loves to clothe itself.

It would perhaps seem plausible then that even if mathematics cannot be defined by its deductions, it might be defined by its processes of deducting, that is, by the systems of reasoning it furnishes.

But mathematics is beyond these things, and its butterfly flights cannot be calculated as so many trajectories in an intellectual space. It is not under the rule of any mechanics, however lax its laws or variegated its forces.

[1] *Amer. Jour. Math.*, 4 (1884), p. 97.

Its wings may rest upon the air, and it may rise because of the sustaining power of matter, but its path none can predict, and the flowers it touches might never be fertilized if its course had passed them by. Says Duhem:[1]

> The faculty of following without fainting lines of long and complicated reasoning, the most minute rules of logic, is not, however, the only one that comes into play in the construction of algebra; another faculty has an essential part in this work; it is that by which the mathematician in the presence of a very complicated algebraic expression perceives easily the diverse transformations permitted under the rules of the calculation to which he can subject it and thence arrive at his desired formulae; this faculty, quite analogous to that of the chess-player who prepares a brilliant stroke, is not the power of reasoning, but an aptitude for combination.

We have passed in review so far the various parts of the subject-matter of mathematics and have found that under no one of them can the whole field be subsumed. Mathematical things are not all expressible in terms of number, or figures, or arrangements, or classes and relatives. Neither can everything be stated in terms of operations, hypernumbers, processes, or systems of reasoning. This is because these things are not only diverse in essence, though they have certain isomorphisms, but because also mathematics consists of more than the entities with which it has to do. For instance, it considers certain aspects of these entities, in their various combinations and transformations. In these aspects there are certain dominant characters, and it might be thought that mathematics can be defined by means of the particular characters in things in which it is interested. These are the central principles of mathematics, and these we will

[1] *Revue des deux mondes* (6), 25 (1915), p. 662.

consider next. They will cut across each of the eight grand divisions of the subject-matter of mathematics. They are the characters of form, of invariance, of functionality, and of inversion. In every investigation of any magnitude in mathematics we would find these four present. They are not expressible in terms of each other, and, as they are not part of the subject-matter itself, they furnish a new basis of attacking the problem of the nature of mathematics.

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CHAPTER X

FORM AS A CENTRAL PRINCIPLE

We have examined the domain of mathematics as to its subject-matter, finding at present eight main divisions into which the things studied may be put. Four of these were found to be of a static type, in which the object studied was, so to speak, a given fixed entity or collection of entities. In the other four we found a dynamic character, the thing studied being transitions rather than states. We found that no one of the eight was sufficiently comprehensive to include the others, and so to become the dominant division and thus furnish a definition for all mathematics. We now have to take cross-sections, as it were, through these eight divisions, along the lines of certain central principles that permeate all of the eight, and which we might think at first would enable us to define mathematics by its chief character, since we cannot do it by its parts. Such attempts have been made. We will consider the central principles to be four: form, invariance, functionality, inversion. The first of these has been suggested as a defining character of mathematics. In his presidential address before the British Association in 1910 Hobson[1] says: "Perhaps the least inadequate description of the general scope of modern Pure Mathematics—I will not call it a definition—would be to say that it deals with form, in a very general sense of the term; this would include algebraic form, functional relationship, the relations of order in any ordered set of entities, such as

[1] *Nature*, 84 (1910), p. 287.

numbers, and the analysis of the peculiarities of form of groups of operations."

This notion was advanced in a series of memoirs by Kempe.[1] He says: ". . . . the conviction must inevitably force itself upon us that, in considering the mathematical properties of any subject-matter, we are merely studying its 'form,' and that its other characteristics, except as a means of putting that form in evidence, are, mathematically speaking, wholly irrelevant."

We meet here a very attractive defining character of mathematics, which at first might seem sufficiently to distinguish mathematics. Many of the peculiarities we have already noticed become intelligible from this point of view. If mathematics studies only form, then the non-material character of its objects and their essentially mental, though not subjective, character become obvious. Form is invisible and intangible, a construction of the mind, yet permanent and not dependent upon the peculiarities of the mind of any one person. The form of a building is realized in stone, but the form existed in the architect's mind before the stone was even quarried perhaps. The form of a symphony is in the musician's mind as much as in the actual performance. If we examine the eight divisions of mathematics, we will find form as an essential character in every one.

Early in the study of numbers questions of form were investigated. The Pythagoreans placed objects in square arrays and studied the "square numbers." They found that two squares put together properly might give a square. They studied the numbers of objects arranged in equilateral triangles, "triangular numbers," and other

[1] *Proc. Lond. Math. Soc.*, 26 (1894), p. 13; *Nature*, 43 (1890), pp. 156–162.

forms. Coming up to modern times, we find numbers studied as to their partitions and their factors. The two notions are combined in the study of perfect and amicable numbers, of which the Greeks knew something. A perfect number is one which has the factors of the number as the elements of one of its partitions. For example, 6, 28, 496, 8128, 33550336, 8589869056, 137438691328, are perfect numbers. In case the factors of each of two numbers are, respectively, the elements of a partition of the other, the numbers are amicable, such as 220 and 284, 17296 and 18416, 9363584 and 9437056. The factorization of numbers of any size admits still only tentative methods of solution. The enumeration of the partitions of a number has been solved in a certain way, which, however, is scarcely practicable of computation for large numbers. The unsuspected forms into which numbers may be cast, thereby permitting their application to the solution of various types of problem, are usually arrived at in the course of another piece of work. A systematic treatment of the cases does not exist. The determination of the coefficients of various types of expansion is another large problem in the consideration of numbers. The expansions may be in Taylor series, or in other rational polynomial expansions, or in rational functions, or may be in terms of given infinite sets of orthogonal functions. Interpolation theorems, the formation of various differences, and the calculus of finite differences belong here also.

In the theory of general ranges the form of the range is of vital importance; the discrete range with no limit points, the ranges with limit points, dense ranges, perfect ranges, the linear continuum, the functional ranges of Hilbert, Fréchet, and others—all have forms which

determine the character of their usage as numbers. In all these the elements are isomorphic to each other.

In the second grand division of mathematics we find a wide variety of elements and an intricate tangle of forms. As elementary elements we usually consider the point, the line, the plane, the ray, the face (which is the plane with an orientation, or a positive face), and the hyperplanes and hyperfaces. The elements of second grade are the point-line (line with a given point on it), the point-ray (ray with a given point on it), the point-plane (a plane through a given point), the point-face (a face through a given point), the line-plane (plane through a given line), the line-face (face through a given line), ray-plane (plane through a given ray), ray-face (face through a given ray). The elements of third grade are point-line-plane (plane through a given line and given point), point-line-face, (face through a given line and given point), point-ray-plane (plane through a given ray and given point), point-ray-face (face through a given ray and given point), and hyperfigures corresponding. Any two or more of these may be combined into an element which is then considered as a new unit. We have in this manner such compound elements as bipoint, tripoint, etc., angle, pencil, polygram, line-cross (two non-intersecting lines in space), line-complex (a set of lines determined by one variable), moulinet (a plane and a point not in the plane), biplane, line-congruence (set of lines determined by two parameters, as, for example, the normals of a surface), sheaf of planes, and many others.[1] When these are combined into the well-known ancient and classic diagrams, as well as the more modern figures, we are bewildered with the wealth of form we must study. We are tempted to admit

[1] See *Encyclopédie des sciences math.*, Tome IV, Vol. 2, IV, Art. 4.

that all of geometry is merely a study of form. They are combined with numerical elements into forms that belong to both the first and second divisions, such as the mass-point, the glissant segment (a segment of given length that can slide upon a line), a vector, a glissant vector, a scale (portion of a plane which can slide in the plane), glissant cycle (portion of a face which can slide in the face), fixed segment, fixed vector, rotatory scale (scale with one fixed point), rotatory cycle (cycle with one fixed point), shear scale (scale with a fixed line), shear cycle (cycle with a fixed straight line), translatory scale (scale with a fixed line and fixed shape), translatory cycle, etc. The modes of combination of these different forms furnish various forms of geometric calculi, one of the simplest of which is Grassmann's Ausdehnungslehre, or science of space. The more modern have been intimately connected with statics.

The third division of mathematics is concerned with the arrangement of elements which may or may not be alike, that is, isomorphic for the arrangement. This was called by Cayley, tactic, by Cournot, syntactic. It is not the same as a study of form. Form is concerned, not with the arrangement, but with properties of the arrangement. If we put together geometric elements, we have an arrangement, while the consideration of the properties of the arrangement is a study of form. If we arrange numbers in magic squares, these are arrangements; the study of magic squares is a study of form. The theory of arrangements considers the possibility of the arrangement, the conditions under which it can exist, its relations to isomorphic arrangements and to arrangements it is a function of, solutions of problems demanding arrangements that fulfil given conditions. The form of the

arrangement is only one part of its study. The study of forms of this kind is often considered to be merely an amusement, but for the mathematician it is a play that stimulates the creative imagination and awakens the creative power, thus bringing about new creatures of mathematics often useful in the most remote regions.

In the fourth division of mathematics, which is concerned with the comparison and study of concepts and relations in general, we are obliged also to study form. This is at once evident when we remember the various types of logical diagrams that have been used to assist the logician. Indeed there is a possibility of reducing the statements of logic of classes and propositions to the consideration of the order of points and other geometric forms. An elaboration of the method is given by Kempe.[1] The conclusions of Kempe are very well worth noting, for they themselves show that it is not possible to reduce all mathematics to the study of form alone. They are as follows:

Whatever may be the true nature of things and of the conceptions which we have of them (into which points we are not here concerned to inquire), in the operations of exact thought they are dealt with as a number of separate entities.

Every entity is distinguished from certain entities, and (unless unique) is undistinguished from others. In like manner every collection of entities is distinguished from certain collections of entities, and (unless unique) is undistinguished from others; and every aspect of a collection of entities is distinguished from certain aspects of collections and (unless unique) is undistinguished from others.

Every system of entities has a definite "form" due (1) to the number of its component entities and (2) to the way in

[1] Kempe, *Proc. Lond. Math. Soc.* 21 (1889), p. 147; *Nature,* 43 (1890), p. 262.

which the distinguished and undistinguished entities, collections of entities, and aspects of collections of entities are distributed through the system.

The peculiarities and properties of a system of entities depends so far as the processes of exact thought are concerned, upon the particular "form" it assumes, and are independent of anything else.

It may seem in some cases that other considerations are involved besides "form"; but it will be found on investigation that the introduction of such considerations involves also the introduction of fresh entities, and then we have to consider the "form" of the enlarged system.

An important element of the subject-matter of exact thought is no doubt pointed out here, but there are certainly other elements that are studied in mathematics. Unless we use the term "form" in such an extended sense that it comes to cover by definition everything in mathematical investigation, we can scarcely include under it such characteristics as those of functionality, invariance, and inversion. When we devise a new system of algebraic numbers, or whenever we add to the existing entities, those others referred to in the last of the paragraphs quoted above, the addition of such entities is within the domain of mathematics and must be accounted for. The theory in question fails utterly to say more than that they are introduced. According to the view advanced in the present treatment, they are usually direct creations of the mind, and mathematics for the most part is the result of the study of such direct creations. Further, the functionality relationships that are added to previous entities are sometimes of more importance than the entities themselves, and it is the functionality that is studied and not the form of the entities plus the functionality. The same may be said of invariance.

The first branch of dynamic mathematics is the theory of operations. It includes the general theory of operators of any type and in particular the theory of groups of operators. The structure of such groups is evidently a study of form. It may often be exemplified in some concrete manner. Thus the groups of geometric crystals exemplify the structure of thirty-two groups of a discontinuous character, and the two hundred thirty space-groups of the composition of crystals exemplify the corresponding infinite discontinuous groups. The study of the composition series of groups, the subgroups and their relations, whether in the case of substitution groups, linear groups, geometric groups, or continuous groups, is a study of form. Also the study of the construction of groups, whether by generators, or by the combination of groups, or in other ways, is also a study of structure or form. The calculus of operations in general, with such particular forms as differential operators, integral operators, difference operators, distributive operations in general, is for the most part a study of structure. In so far as any of these is concerned with the synthesis of compound forms from simple elements, it is to be classed as a study of form, as the term is here used.

In the study of hypernumbers much of the work has so far been only that of constructing certain algebras, that is, numbers which have as qualitative units given hypernumbers, are built up in the most general way and their combinations examined. These combinations furnish the laws of the particular algebra. These are often stated in a multiplication table of the algebra. The sub-algebras and their relations to each other, composition series, and like questions also belong in this division. It is the invention of these hypernumbers, Kempe says, that has

enabled the geometer to simplify his problems by their adjunction, the introduction of the new elements making it possible to restate the geometric problem. We might add that the whole of mathematics consists in adjoining to the data of experience those elements that human thought has created in order to simplify the problems of the data of experience.

The third division of dynamic mathematics is that of processes. The simplest processes are those that pass from a set of objects on a single range to a set on a single range. Thus the process of differentiation passes from a set of continuous functions to a set of functions possibly also continuous. The process of expansion in fundamental functions passes from a single range of functions to a range of series whose coefficients are to be determined. The process of variation of an ordinary function may be viewed as a process from a range which has infinite manifoldness to a range with infinite manifoldness. Every function may be looked upon as a mode of transition from a range or set of ranges to a range or set of ranges. And every transmutation of a function into another may be viewed as a transition from one mode of transition to another mode of transition.

The last division of dynamic mathematics is that of systems of inference. The form of a demonstration, the structure of a proof, would seem almost to be the vital part of the reasoning. If we describe the manner of building a demonstration, what more is there to say?

However, a little reflection will show that these various structures, whether of number or of inference, are only one of the characters that mathematics considers. Besides these it takes into account those features that are invariant under a change of form. These we will consider in

some detail in the next chapter. It is enough to notice here that there are such invariants. Further, a large part of mathematics is concerned with the correspondences that exist between structures and which are called functions. The whole theory of functionality is concerned, not particularly or directly with structure of the entities, but with the properties of the functional correspondence. Finally, most mathematical investigation leads rather abruptly to the solution of certain questions calling for the existence mathematically of entities with assigned properties. This is the theory of equations or, more generally, of inversions. Questions of structure or form are not the prominent thing in these.

It seems evident enough, therefore, that mathematics cannot be reduced to propositions about form alone, at least unless we include under form other characteristics than those that relate to structure alone. And we should not strain the meaning of a term in order to make it available for a definition. Form is a definite term to apply to one of the characters with which mathematics is concerned, and we will restrict it to that use.

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Kempe, "Theory of Mathematical Forms," *Phil. Trans. Roy. Soc. Lond.* (1886), Part I, pp. 1–70.

CHAPTER XI

THEORY OF INVARIANTS

The principle of invariance has also been considered to be the essence of mathematics. In his presidential address before the British Association in 1901 MacMahon[1] said: "It is the idea of invariance that pervades today all branches of mathematics." And Forsythe[2] in his presidential address in 1897 said:

> The theory of invariants has invaded the domain of geometry, and has almost re-created the analytical theory; but it has done more than this, for the investigations of Cayley have required a full reconsideration of the very foundations of geometry. It has exercised a profound influence upon the theory of algebraic equations; it has made its way into the theory of differential equations; and the generalization of its ideas is opening out new regions of most advanced and profound functional analysis. And so far from its course being completed, its questions fully answered, or its interest extinct, there is no reason to suppose that a term can be assigned to its growth and its influence.

In his *Continuierliche Gruppen*, edited by Scheffers, Lie says (p. 665): "In recent times the view becomes more and more prevalent that many branches of mathematics are nothing but the theory of invariants of special groups." We need therefore to examine this idea in detail to ascertain its precise domain.

In the theory of ranges we find a large part of number theory is devoted to the study of quadratic forms, and

[1] *Nature*, 64 (1901), p. 481. [2] *Nature*, 56 (1897), p. 279.

afterward to the study of forms of any order. Their invariants and covariants are of the highest importance and are the basis of most of the investigations. We may classify here also as invariants the theory of congruences, which, with the theory of arithmetic forms, makes up the greater part of the theory of numbers. Further, the modular geometries of Dickson belong to the field of invariants, in the general sense we give the term.

When we come to the theory of manifolds, we enter a region first reduced to order by the theory of invariants. The invariants, covariants, contravariants, and other forms of invariantive character for the binary quantics, ternary quantics, and quaternary quantics only a few years since furnished heavy courses in most universities. That they have disappeared as titles merely means that they occur under other titles, with no overemphasis. This branch of algebra and analytic geometry is so important that it once threatened to displace the other parts. In many texts it even was given undue prominence at the expense of other topics. On the algebraic side, determinants, symmetric functions, and related branches belong under this head. We must include the whole of synthetic geometry, which is the study of invariants, or at least may be so regarded. Moreover, we have the congruence theory of algebraic forms, called the modular theory, of the highest importance in algebra, and, when combined with the idea of domain of integrity, yielding an intimate knowledge of the structure of algebraic expressions. It is the foundation of Dickson's theory of abstract fields and thus defines finite ranges that are of great usefulness.

In the third division of static mathematics we do not find as yet very much progress in the study of invariants.

The theory of isomorphism in arrangements can be placed properly here as well as problems of transitivity. The whole of this branch is too undeveloped to expect much knowledge of the invariant characters it may have.

The logic of classes, relatives, and propositional functions in general possesses few invariants that have been systematically developed. The rules of the calculus constitute about the only logical invariants known so far, although recent investigations are drifting this way.

When we come to the field of operators, we find a rich harvest of invariants. It is sufficient merely to mention projective geometry, with regard to which Steiner[1] said: "By a proper appropriation of a few fundamental relations one becomes master of the whole subject; order takes the place of chaos, one beholds how all the parts fit naturally into each other and arrange themselves serially in the most beautiful order, and how related parts combine into well-defined groups. In this manner one arrives, as it were, at the elements, which nature herself employs in order to endow figures with numberless properties with the utmost economy and simplicity." We notice geometric transformations in general, of which Lie[2] said: "In our century the conceptions of substitutions and substitution group, transformation and transformation group, operation and operation group, invariant, differential invariant, differential parameter, appear more and more clearly as the most important conceptions of mathematics." We must not leave out analysis situs, the study of continuous one-to-one transformations, such transformations as can happen to a rubber surface or to a battered tin can. This is the most fundamental of all the

[1] *Works*, 1 (1881), p. 233.

[2] *Leip. Ber.*, 47 (1895), p. 261.

branches of geometry, its theorems remain true under the most trying conditions of deformation, they come the nearest to representing the necessities in an infinite evanescence that any theory can furnish. If we were to add to it a new analysis situs of an infinitely discontinuous character, we might hope that some day we could furnish certain laws of the natural world that would hold under the most chaotic transformations. If we increase this already tremendous list with the grand theories of differential and integral invariants, we can almost feel ourselves the masters of the flowing universe. We find ourselves able to see the changeless in that which is smaller than the ultra-microscopic and also to ride on the permanent and indestructible filaments of whirling smoke wreaths throughout their courses to infinity. Wars may come and go, man may dream and achieve, may aspire and struggle, the aeons of geology and of celestial systems may ponderously go their way, electrons and dizzy cycles of spinning magnetons, or the intricate web of ether filaments may "write in the twinkling of an eye differential equations that would belt the globe,"[1] yet under all, and in all, the invariants of the mathematician persist, from the beginning even unto the end.

In the branch of hypernumbers the list of invariants is not extensive as yet. The automorphisms of an algebra, however, are necessary for the investigations of its structure and of its applications. The invariant equations of an algebra define it and also show to what things its numbers belong naturally. This field will become as large in time as that of algebraic invariants is now.

The invariant theory of the branch we called processes is not touched. Nor is the invariant theory of schemes

[1] Herschel, *Familiar Lectures on Scientific Subjects*, p. 458.

of inference yet investigated. When it is developed, we may really talk about laws of thought.

Just as in the principle of form we are studying chiefly the synthetic character of mathematics, so in the principle of invariance we are studying the permanent character of mathematical constructions. Its results are everlasting, and we have in them a growing monument to the human intellect. But we cannot afford to confuse the determination of invariants in mathematical constructions with the whole of mathematics and with the permanent character of mathematical theorems. In other words, the theorems of mathematics are the invariants of the field of mathematical investigation. Among these are theorems regarding the invariants of some of the objects of investigation under transformation. Mathematics contains many theorems which are invariants of thought, but are not theorems about invariants of mathematical objects. The fact that every mathematician comes to the same conclusion with regard to the same subject of investigation shows the invariant character of the intellect. The subject of investigation itself, however, need not be a study as to invariancy, but anything in the realm of mathematics. For instance, problems as to the theory of functions may not deal with invariants at all. Of course this is the same as saying that the questions of the mathematician are not always questions as to the permanency of something, but may be questions as to synthetic construction, as to correspondence, or questions as to the solution of equations of various types.

The theory of invariants is evidently one of the central principles of mathematics, yet mathematics cannot be reduced to mere problems of invariance. The invariants of mathematical objects serve to characterize them, but

not to define them completely, nor do they give other properties of that which they define. These must be sought for along the lines of the other central principles.

We find in the invariants of mathematics a source of objective truth. So far as the creations of the mathematician fit the objects of nature, just so far must the inherent invariants point to objective reality. Indeed, much of the value of mathematics in its applications lies in the fact that its invariants have an objective meaning. When a geometric invariant vanishes, it points to a very definite character in the corresponding class of figures. When a physical invariant vanishes or has particular values, there must correspond to it physical facts. When a set of equations that represent physical phenomena have a set of invariants or covariants which they admit, then the physical phenomena have a corresponding character, and the physicist is forced to explain the law resulting. The unnoticed invariants of the electromagnetic equations have overturned physical theories, and have threatened philosophy. Consequently the importance of invariants cannot be too much magnified, from a practical point of view. But for the pure mathematician there are the other phases that must also be considered and which are important. The theory of invariants, like the theory of form, is not the most important theory in mathematics—that high place is reserved for the theory of solutions of equations of all kinds.

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CHAPTER XII

MATHEMATICS AS THE THEORY OF FUNCTIONS

In his lectures on the development of analysis, at Clark University, Picard says:

The whole science of mathematics rests upon the notion of function, that is to say, of dependence between two or more magnitudes, whose study constitutes the principal object of analysis. It was a long time before account was taken of the extraordinary extent of this notion, a circumstance which was very happy for the progress of the science. If Newton and Leibniz had thought that continuous functions need not have derivatives, which is in general the case, the differential calculus would not have been born, likewise, the inexact ideas of Lagrange upon the possibility of developments in Taylor's series rendered immense service.

When we look over the range of modern analysis or the differential and integral calculus, with all its applications, including the functions of complex and other variables, we may at first consider that it would be safe to define mathematics as the whole theory of functions in general. We shall inspect the field, however, a little more closely, remembering that, in order to classify any branch of mathematics under the theory of functions, it must deal with the idea of dependence or correspondence.

We meet in the theory of general ranges, first of all, functions that are determined by the assignment of a finite number of numerical values. These are of little interest in the present discussion. Then come functions that run over a denumerably infinite range of values.

Following these, we have the theory of functions of a real variable over any range, usually a point-set. This includes, of course, arithmetic and algebraic functions, but it also takes in functions in which the dependence of the one variable on the other is determined in any manner. If the range admits of the notion of continuity or discontinuity, we include both classes. The ordinary infinitesimal calculus, so far as it deals with one independent variable, enters here. The method of definition is not material, whether by expression as series or expansions of various types, or by definite integrals, or by limits, or by artificial laws. It is by means of the theory of functions of real variables that we have reached the point from which our perspective is corrected, and we can see most easily perhaps that mathematics is not dependent wholly upon intuition in the usual sense. For the study of functions of a real variable has produced the continuous function that has no derivative, which we cannot study intuitively at all; the function which correlates the points of a square to the points on a line; the curves that fill space full; and has, indeed, so modified the conception of what it is that we study in geometry, that we almost assert that, whatever it is, it surely is not space. The theory has also produced the means of exhibiting in calculable form various expressions for what are called arbitrary functions. The original notion of function meant, of course, little more than a single law by means of which the values could be found. A single law meant a single expression, and more than a single law meant more than one expression. But we passed that stage a century ago and can now represent a function which is given by a great number of laws, also by a single expression. We must also include here the study of functions over ranges

that are not representable as point-sets, with the consequent changes of idea as to continuity, etc. This is one part of general analysis. The invention of general ranges is due to the demands of functional analysis, as general analysis is due to the increasing wealth of functions and the necessity of classifying them and discovering their essential properties.

In the region of vectors, or geometry, or manifolds, we have the whole theory of many real variables. In partial derivatives and multiple integrals the calculus appears, the field of application becomes an enormous one. It includes also what may be termed vector-fields of all the known types. We are herein beginning to approach mechanics and physics. If the vector-fields are fields of force, or velocity, or acceleration, we are in the region of mechanics or electrodynamics. If they are fields of stress and strain, we are in the region of elasticity. If they are velocity fields and fields of deformation, we are in what is called hydrodynamics. Further, we must include here an invention of the twentieth century—functions of lines, surfaces, hypersurfaces, hyperfigures in general, and functions of functions, indeed the whole of modern functional analysis. We see at once that any kind of dependence that can be determined by the dependence of numbers on numbers can be caught in this mighty machine and handled with perfect ease. The chief aim of the more exact sciences is to arrive at the statement of laws in just this way. For our purposes, however, we must notice that in the development of these general subjects many new concepts and even new mathematical methods have had to be devised. We must include, also, differential geometry, at least so much as does not have to do with groups. We may, it is true, put many other develop-

ments of mathematics into the study of functions as introductions, but, while they may be classed thus as parts of the theory, just as we find the theory of ensembles usually so placed, yet in a philosophic analysis of mathematics they are not investigations into functionality and should be classed separately.

In the region of tactic we have as yet only a few developments that could be called functional. In the theory of classes and relatives, the propositional function has become very prominent recently and is properly the beginning of the logical theory of function.

In the dynamic phase of mathematics we meet again very large developments on the functional side. In the theory of operators we have all the geometrical transformation groups, not in themselves, nor in their invariants, but as operating upon geometric figures. The homographic, conformal, reciprocal, and other transformations, the representation of surfaces upon other surfaces—in all these cases we have a right to count the result of the transformation a function of the transformed entity.

So, too, in the consideration of groups themselves we may consider one group to be a function of another when it is derived from the other. If a function is produced from another by a transformation which can be applied to a class of functions as arguments of the transformation, like differentiation, for instance—what has been called by Bourlet a transmutation—then we must list it here. The whole theory of functional operators which convert functions into functions belongs here also. Included in this, we find general integral equations. Thus the field is seen to be very comprehensive. But again it is marked by the invention of many new concepts as well as methods.

When we consider the functions of hypernumbers, we meet at once a branch of mathematics which overshadows many of the others we have named, the functions of a complex variable. It was born when Cauchy discovered the integral theorem

$$\int_c f(x)dx=0,$$

published in 1825. On this integral theorem he founded that method of studying these functions which bears his name. Riemann founded the theory later upon differential methods, and geometric intuition; and Weierstrass, upon the method of infinite expansions; but the three have been combined into one magnificent and symmetric theory. It possesses a completeness that the theory of real variables does not, inasmuch as a function that is defined for any continuous set of points can be defined, that is, expressed, wherever it exists, provided it is analytic. Since the real axis is part of the complex plane, many of the theorems can be made to apply to the theory of real variables.

The functions of quaternions and other hypernumbers have not been very much studied as such, although much of the work on functions of many variables can be interpreted as applying here, except that the character of the hypernumber is not in that case apparently present. Functions of operational fields and general function theory belong here also. Functions of processes have not been developed, nor functions of deduction.

We see easily from the preceding rapid survey of the territory of theory of functions that, while it is very large and important, it does not include the regions of the theory of form or invariance. In only a small degree may these be considered to treat of functionality. Functionality is

simply one of the great central principles of mathematics. But the evolution of the notion of function from the small beginnings in which only the integral powers of numbers were the functions handled, up through all the developments into algebraic functions, transcendental functions, functions of many variables, to the modern idea that one figure is a function of another, that a class of functions may be a function of another class is a very instructive study of the evolution of mathematics from simple notions of little complexity or subtlety up to notions that are not only very delicate in their distinctions, but quite intricate in their inner structure. In the course of such development have arisen the notions of continuity, discontinuity, uniformity, relative uniformity, closure, linearity, distributivity, dominance, derivability, and numerous others every one of which is the result of the creative elaboration of the relations visible or conceivable in the subject-matter.

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CHAPTER XIII

THEORY OF EQUATIONS

One of the chief sources of mathematical advance is the consideration of problems. We do not say the *solution* of problems, for frequently the problems are not solved, indeed, may not be solvable, yet in their consideration the mathematician has been led to invent new methods, new concepts, new branches of mathematics. One of the sources of problems, from which flows a growing stream as knowledge progresses, is natural science. We need only remember the famous problem of three bodies and the attendant series of memoirs on mathematics which has been its outcome. The equations of mathematical physics have suggested many widely diverse branches of mathematics. Problems in geometry suggest theorems in arithmetic, and problems in arithmetic have suggested geometrical advance. But by far the largest number of problems emanate from the mind itself in its study of mathematics. The inventions of the mathematician bring a constantly growing number of problems which in turn suggest a still wider field of investigation. Says Hilbert:[1] "If we do not succeed in solving a mathematical problem, the reason frequently consists in our failure to recognize the more general standpoint from which the problem before us appears as a single link in a chain of related problems. After finding this standpoint, not only is this problem frequently more accessible to our investigation, but at the

[1] *Bull. Amer. Math. Soc.* (2), 8 (1902), p. 443.

same time we come into possession of a method which is applicable to related problems."

The chief point of interest to us in the present investigation is, however, the creation of new ideas—new entities, we may well say—in mathematics, from the attempts to solve problems. These we will look at in detail.

One of the first attempts to solve problems was that of the solution of what we call diophantine equations, which have to be solved in integers. At present this kind of problem culminates in the famous last theorem of Fermat. In the attempt to prove this, Kummer was led to invent the ideal numbers which he used, and which in turn lead to the general theory of algebraic numbers and algebraic integers. We find created here the branch of mathematics called higher-number theory, as well as the introduction of the various domains of integrity in which we intend to work. Along the same line is the Galois theory of equations, which consists in finding the domain of integrity in which the solutions or roots of a given equation lie. On the one hand, we find from these problems the notion of rationality and, on the other, the notion of hypernumber, springing up spontaneously. The negative and the imaginary owed their origin to the necessity of finding solutions for certain equations. They were not known for centuries afterward to exist in nature in any way, and neither were they objects of intuition in any ordinary sense of the term. We might call them products of that faculty denominated by Winter, the transintuition, which is, so to speak, the intuition of the pure reason alone.

In the same region of solutions of problems arising from a single range, we have the list of functions invented to solve the ordinary differential equations of a single

independent variable. Hyperbolic functions, elliptic functions, hyperelliptic functions, Abelian functions—these were invented, as well as hosts of others, in order to complete the solution of the differential equations that arose in the course of the work of the mathematician. They in turn brought up the question of a functional domain of integrity, that is, a study of the conditions under which such differential equations could be solved in terms of given functions, as, for example, when a differential equation can be solved in terms of algebraic functions, circular functions, elliptic functions, etc. This is the Picard-Vessiot theory, similar to the Galois theory of equations and dependent upon the groups of the differential equations. The creation of new functions which are derived not from experience, but from their properties, is a sufficient phenomenon in itself to prove the autonomy of mathematics and its self-determination. Further, we find in the recent developments of difference-equations an opening of the new field which will lead to further inventions.

The theory of differential equations of several independent variables is responsible for the invention of spherical harmonics, ellipsoidal harmonics, harmonics in general, and a wide variety of unnamed functions. We also have the enormous list of solutions of equations with total differentials, which lead to functions that are not easily representable in the ordinary way. The greater part of mathematical physics lies in this region, since mathematically it is merely a consideration of the solutions of differential equations. The invention of the Green's functions alone and the expansion of this notion to cover a large class of functions of several variables, which are defined by differential equations with given boundary

conditions, is another branch of mathematics quite capable of demonstrating the fertility of the mind.

We need not stop to consider the solutions of problems of construction or problems of logic. They have their place, and what small synthetic character logistic has, lies in its few contributions in this direction. We find in the forms of atoms, molecules, and multimolecules ideals of mathematical chemistry.

In the theory of operators we have the invention of the automorphic functions as the functions which are solutions of certain equations of operators, particularly operators that form groups. The periodic functions, the doubly periodic functions, and others have extended mathematics very far. The study of integral equations, which is properly the study of functions that satisfy certain operational equations of a linear character, or to a small degree non-linear character, has introduced, not only new methods and new solutions, but a new point of view for the treatment of a wide range of mathematics. It enables us to define orthogonal functions in general and suggests other functions than the orthogonal, which remain for the future to study. Closely following it is the theory of functional equations in general, in which we undertake to find functions as the solution of certain functional equations. This includes the theory of operations and leads up to a theory we may call functional analysis. The calculus of variations belongs here, one of the oldest branches of mathematics of this type. Many problems in physics may be stated as problems in the calculus of variations, indeed, this method of statement seems to be the most unifying we have today. The determination of the solutions of variational questions is one of the important divisions of functional analysis.

The solutions of problems arising in a similar way from the functions of complex variables are intimately connected with the preceding forms, and usually little distinction would be made between them. However, the problems involving functions of several complex variables have peculiarities that must be taken into account. The problems arising in the consideration of functions of quaternions have yet to be investigated, and, when they have been studied in full, they will no doubt lead to many new ideas.

Problems in games such as the endings in chess are so far only amusements, and problems in the solutions of questions of deductions are purely in a tentative state. When mathematics has devised methods of producing the theories of scientific elementary ideals, the progress of science will be rapid. And these solutions will come in time, for all science is approximating a mathematical statement. The methods of science and those of mathematics are practically the same, and this identity will be revealed more plainly as the advances of mathematics enable us to handle problems of deduction.

This is the most important central principle of mathematics, namely, that of inversion, or of creating a class of objects that will satisfy certain defining statements. If the mathematician does not find these at hand in natural phenomena, he creates them and goes on in his uninterrupted progress. This might be considered to be *the* central principle of mathematics, for with the new creation we start a new line of mathematics, just as the imaginary started the division of hypernumbers, just as the creation of the algebraic fields started a new growth in the theory of numbers. Thus it is evident that mathematics is in no sense a closed book; that its chief concern

is not to solve problems that arise for the engineer or the scientist; that it has no absolute, whether an absolute space, or an absolute time, or an absolute number, or an absolute logic, which is the bound of its development. Mathematics has thus been able to answer the philosopher's question definitively in the negative: Is there an absolute which we may never hope to compass in any way? We face thus an infinite development, an evolution that never closes, that meets its obstacles by an act of creation which leaps over them. And in giving the mind its freedom here it has given it its freedom in science and philosophy both. It has guaranteed the progress of human thought throughout all the ages to come.

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CHAPTER XIV

SOURCES OF MATHEMATICAL REALITY

We have passed in review the subject-matter of mathematics and the predominant characters of the objects with which mathematics concerns itself. We found that the objects it studied were numbers, figures, arrangements, propositional functions, operations, hyper-numbers, processes, and deductive systems. The chief characters of these objects with which it busied itself were those of structure, invariance, functionality, and inversion. No one of these different classes of entities or characters, however, could furnish a satisfactory definition of mathematics which would include the entire subject. Yet we find in mathematics that subject whose results have lasted through the vicissitudes of time and are regarded universally as the most satisfactory truths the human race knows. We have still to inquire the source of the reality that is in mathematics, its methods of arriving at truth, and the realm of validity of its conclusions.

The reality in mathematics has been ascribed by some to its experimental character; in short, it has been regarded as a branch of physics. We find indeed a method given by Archimedes, and considered by him to be of great use, in discovery at least if not in rigor, for finding areas and volumes by mechanical considerations. While this would not be exactly an experiment, it is at least an ideal experiment, and could be carried out in fact in material of a proper kind. Archimedes[1] arrived at the

[1] See Milhaud, *Nouvelles études sur l'histoire de la pensée scientifique*, p. 135.

area of a parabolic segment by considering it in equilibrium at the end of a lever, at the other end of the lever constructing a triangle, whose area is therefore the same as that of the segment. We remember also that Galileo cut out of tin foil a cycloidal arch and found its area to be three times that of the generating circle. More recently kinematic methods have been applied to various problems in geometry. These few facts, however, are not of so much importance for their results as for the tacitly received principle that all the results of mathematics, whether of a physical origin or not, could nevertheless be made the subject of experiment, and the theorems should prove to be true within the limits of error of the measurements. This ascription of the reality of mathematics to a physical foundation is a positivistic explanation of the truth of mathematics. Those who hold the view strongly even go so far as to consider all mathematical results that cannot be so examined—like the theorems of four-dimensional space, for instance—to be purely of symbolic interest and only possibly, not absolutely, true. The latest exponent of this philosophic standpoint is Enriques.[1] He considers geometry to be a system of concepts which have been extracted from sense-data, somewhat like composite photographs whose vagueness has been replaced by sharp-cut features rather arbitrarily chosen. These concepts are put together by certain observed relations, according to the particular sense-data concerned. Thus, for instance, from sight we derive relations of projective geometry, from touch the relations of metric geometry. On the basis of these idealized features of the physical world a body of theorems has been worked out which apply to the world only so

[1] *Problems of Science.*

far as the idealized data really fit the world. The theorems must be verified by the success of their applications directly and indirectly. All that can be said of such a system of deductions as the Lobatchevskian geometry is that its conclusions are consistent with each other so far as we know, and its truth can only be verified if we can find some way to interpret its theorems in ordinary Euclidean terms, as Poincaré has done. The Lobatchevskian system becomes thus a purely symbolic or empty form, and its words have no meaning. It is much the same as if in dynamics we study the motions of a planet under a force varying inversely as the seventh power of the distance. There is no reality back of the study, and its interest is purely artificial.

It may be said in reply to all such arguments that the real world referred to for the ultimate test of reality is after all indisputably the world as we know it, not as we do not know it. The greatest hypothesis of all is that there is a so-called objective world, unless by the latter is meant only that view of what we know as the world which regards it from the standpoint of certain hypotheses that most sane men agree upon. For example, most persons who have reflected upon the matter agree that it is simpler to suppose that the earth rotates than that it is stationary. More facts can be arranged under fewer laws in this way. Nothing whatever in our sense-data tells us that the earth is rotating. We may interpret the same sense-data from the viewpoint of the hypothesis that says the earth does not rotate. Millions of men actually have so interpreted their sensations. No so-called proof that the earth rotates does more than produce some phenomena that we would expect it to produce if it did rotate. But the veriest tyro in logic knows that one cannot argue

that a premise is true because the conclusion is true. Now the feeling of certainty in mathematics is deeper than would be the case if it depended only upon such a basis as the one stated above. No conceivable experiment can prove or disprove the theorems of geometry, for in the first place they are not at all theorems about material objects, but about purely immaterial things. No mathematician imagines for a moment that his triangles are wooden or steel. They are wholly mental constructions, and thus beyond the reach of experimentation of the physical kind. Such experimentation may show how far the properties of such figures fit in with the properties we use for the organization of our sense-data from such objects. Whether, for instance, we can reconcile the use of non-Euclidean geometry and the rectilinear motion of light might be a question, but whether light has a rectilinear path or not we shall never know definitively, while we do know definitively the properties of figures in non-Euclidean space. Just as it is simpler to keep our present list of forces and laws, and to suppose that the earth rotates around the sun and revolves upon its axis rather than to suppose that it is stationary, and to set up a new set of laws which would be much more complicated, so it may be simpler to suppose that light moves in Euclidean straight lines and that figures are most easily handled that way. That one can ever say Euclidean space is true or false for the physical world is a chimera. The source of mathematical reality is not in the sensory world. It is mathematical reality on the contrary that transcends the sensory world by studying imaginary worlds, and it is mathematics indeed that organizes the sensory world and makes it intelligible.

Says Milhaud:[1]

Mathematics rouses the interest and enthusiasm of philosophers because, consciously or not, everyone feels that it realizes the miracle of assuring most clearly its success less by a docile submission to the reality that offers itself to us than by the spontaneity of the outbursts of the mind, by the richness and the power of its creative activity. The miracle is such that it necessarily encounters skeptics, and we must consider it for an instant.

Let us make all the concessions possible to those who are disposed to deny it. Let us accept, if they desire it, every suggestion of experience, at the base of the mathematical sciences, in the notions of number, magnitude, quantity, space, movement, line, surface, volume, variation of velocity, infinitesimal increment, limit, etc. (without even asking if there is not at least a little truth in the critical theories, and if in these first notions there does not enter some formal necessity from the human mind); let us accept also all the solicitations which, in the course of the development of analysis, geometry, and mechanics, come incessantly from the ever new difficulties of the problems that nature sets; let us not deceive ourselves, as no doubt they will ask us to do, as to the work of elaboration which quite naturally our mind performs upon the data of experience, when it generalizes or abstracts in such wise as to construct a picture out of permanent images and words which serve to designate them, but yet without indeed there arising any question of a special creation—it remains incontestable that neither these data nor the current operations suffice to furnish the veritable elements which the mathematician handles. These latter, far from being the residues of experience, are formed by an incessant effort to eliminate from the image all that retains any concrete and sensible quality.

A continuous transformation transports the mathematician from the conditions which hem in every intuitive view, and

[1] *Nouvelles études sur l'histoire de la pensée scientifique*, p. 22.

permits him thus to give birth to those creatures of the reason which his intellect dominates and by whose aid it forges endless chains of propositions which rigorously imply each other. Ultimately, beyond any visible external stimulus, by a kind of natural current of thought, problems set themselves, definitions call forth new definitions; generalizations of a special character extend at every moment the domain of validity of a notion and enlarge as well the field open to rational constructions; so that in the presence of a treatise on analysis, or equally on geometry, one is astonished by the richness and variety of a whole world of conceptions which seem evoked by the magic power of the mind from the initial data, accepted once for all. Shall we say that it is an illusion, and all these new creations translate really only data borrowed from experience, or from a sort of sense-intuition which is latent in it? Possibly, but beyond reducing everything to mere suggestions, beyond the fact that the sensuous intuition is inseparable from the power of refining and combining in every way the elements it furnishes, beyond the fact that experience itself often takes a special character which forbids its being exterior, as when it is determining the form of an algebraic expression—is it not still obvious that the endowment which a new definition has, is not indispensable for the enrichment of mathematics, and that long chapters on analysis or geometry, where theorems upon theorems, constructions upon constructions, are amassed, exhibit an example of developments manifestly unlimited, without the addition of any new notion to those with which one began? Shall we say then that the mathematician only draws out of the initial data what was implicitly contained? That would be a mere mode of speech, for who does not appreciate all the activity, all the genius, all the creative power necessary to see and to bring forth what is hidden in the initial ideas, or, more exactly, shall we not say, to realize on them as a foundation the very richest constructions?

Another source to which the reality of mathematics has been ascribed is to a non-material world, a world

of universals. We are spectators and students of this world, but it is external to us, even if non-material; its structure exists outside of, and independent of, our thinking faculty or our existence. We discover its eternal verities, but they were existent before us and were true throughout all time, and will be true throughout all time to come. This is a reduction to a realism of the most absolute type. It is present in Plato's philosophy in a somewhat obscure but sublime form, and of late has appeared in the philosophy of Bertrand Russell. Plato insisted that the square and the diagonal drawn on the sand were merely things that resembled the real square and the real diagonal, which could be perceived only by thought. He taught that beyond ordinary mathematics, as far as it is beyond the physical mathematics, there is a mathematics which has for its objects the world of ideal numbers and ideal figures. Numbers are ideas. To each of the natural numbers up to ten there corresponds an idea, and each of these ideas has its own form not derived from the mere juxtaposition of units. We are none too certain as to just what he had in mind, but from a study of the books "M" and "N" of the *Metaphysics* it would seem that he must have had a glimmering of the general notion of ensemble and of operator, since he makes much of the dyad which is a couple and the dyad which is a duplicator. The tetrad, for instance, was a combination of the two dyads. An attempt was made also to identify the number-ideas with the figure-ideas. For instance, the line corresponded to the dyad, the surface to the triad, etc. On the basis of notions such as these an attempt was made to build up an absolute philosophy which had for its object of study ideas in general.

The more modern statement of a similar view is to be found in the philosophy of Russell. We may quote Keyser's[1] statement of it:

That world, it is true, is not a world of solar light, not clad in the colors that liven and glorify the things of sense, but it is an illuminated world, and over it all and everywhere throughout are hues and tints transcending sense, painted there by the radiant pencils of psychic light, the light in which it lies. It is a silent world, and, nevertheless, in respect to the highest principle of art—the interpenetration of content and form, the perfect fusion of mode and meaning—it even surpasses music. In a sense it is a static world, but so, too, are the worlds of the sculptor and the architect. The figures, however, which reason constructs and the mathematic vision beholds, transcend the temple and the statue alike in simplicity and in intricacy, in delicacy and in grace, in symmetry and in poise. Not only are this home and this life, thus rich in aesthetic interests, really controlled and sustained by motives of a sublimed and supersensuous art, but the religious aspiration, too, finds there, especially in the beautiful doctrine of invariants, the most perfect symbols of what it sees—the changeless in the midst of change, abiding things in a world of flux, configurations that remain the same despite the swirl and stress of countless hosts of curious transformations.

But at the same time we are brought face to face with a most startling conclusion. If there is a world of entities that are supra-sensible and yet transcendentally absolute, and if our propositions in logic and mathematics are mere registers of observations of these entities, then we must admit that contradictions and false propositions and error in general exist in exactly the same sense and the same way as the true propositions and stable constructions. The noxious weeds of falsehood, inconsistency,

[1] "The Universe and Beyond," *Hibbert Journal*, 3 (1904), p. 313.

and evil in general grow side by side with the good, the true, and the beautiful. The Devil sows his seeds and raises his harvests along with the God of truth. "No true proposition could be called false. As well say that red could be a taste instead of a color. What is true, is true; what is false, is false; and there is nothing more to say."[1] And in our observations how are we to know the true from the false? Think of it! An eternal world in which everything is equally real, equally stable, equally important, full of both the true and the false, the possible and the impossible, that which is and that which never was, rank with error even though full of harmony! What criterion can differentiate between the two? What Virgil can guide our shrinking minds to Paradise through such an Inferno?

There is no problem at all in truth and falsehood; . . . . some propositions are true and some are false, just as some roses are red and some white; . . . . belief is a certain attitude toward propositions, which is called knowledge when they are true, error when they are false. But this theory *seems* to leave our preference for truth a mere unaccountable prejudice, and in no way to answer to the feeling of truth and falsehood. . . . . The analogy with red and white roses seems, in the end, to express the matter as nearly as possible. What is truth, and what falsehood, we must merely apprehend, for both seem incapable of analysis. And as for the preference which most people—so long as they are not annoyed by instances—feel in favor of true propositions, this must be based, apparently, upon an ultimately ethical proposition: "It is good to believe true propositions, and bad to believe false ones." This proposition, it is to be hoped, is true; but if not, there is no reason to think that we do ill in believing it.[2]

[1] Russell, *Congrès inter. de phil.*, 3 (1901), p. 274.

[2] Russell, *Mind* (new series), 13 (1904), pp. 523–524.

Let us hope that the sum of the three angles of a triangle is 180°, for at least it does not damage us any at present to believe it! What a travesty of truth! Is it remarkable that logistic philosophy collapsed in its own contradictions? And were Poincaré living he still would be waiting its successor.

(3 A third source to which the reality of mathematics has been ascribed is one from psychology, the structure of the mind itself as a static entity furnishing the data and conclusions of mathematics. Mathematics is, from this standpoint, only a statement of laws of mind, just as physics is a statement of the laws of the natural world of inanimate objects. We may pass over the history of this idea to Kant, who is to be considered as the great exponent of it. For example, number is not a concept, he says. It is rather a mental scheme by which an image can be constructed for a concept. It is, so to speak, a sort of working drawing for the mental activity, according to which any particular number may be visualized. The number 100, for instance, is not a composite photograph of the different centuries we may have witnessed in our counting of objects, nor is it a symbol which is to be filled in, like the countersigning of a check, by reality, having no validity till properly filled in. It is rather an innate mode of constructing a hundred objects of whatever matter comes handy. Number, indeed, is the unique scheme by which the chaos of data of the senses is synthetized into homogeneous wholes. This schematic ability, he says, is hidden deep in the mysteries of the human soul, and it is difficult to exhibit its true nature to the eye. A few examples from modern mathematics may make his meaning clear. In his sense, for instance, the rational numbers, or the rational points, say, from

0 to 1, are a disorganized set, a chaos, no unity visible in them. Hereupon the scheme which has been called density appears, an a priori notion of the mind, and the rational numbers are unified under the term "dense set." Further, no analysis of the individual points would ever have revealed this term dense, since it does not belong to the individual points, and appears only when they are synthetized by means of this purely mental scheme. In other words, the points themselves are not dense, it is the unified collection, a mentally unified collection, which is dense. The notion dense is not analyzed out of the collection any more than out of the individuals; it is put into the collection by the mind itself. As another example, we might take uniform convergence. Given a series of terms in functions of x which defines a function of x, then there exists uniform convergence for the series if the following condition is fulfilled: stating the property a little roughly, draw the graph for the function defined, and draw a parallel curve on both sides of the graph, making a strip of any constant arbitrary width; the series has uniform convergence if the approximation curves beginning with some determinate curve, say the n-th, given by the first term of the series, the first two terms, the first n terms, and so on, lie entirely within the strip. Now no one of the curves drawn can have the property of uniform convergence. This property would never have entered our discussions in case only individual curves had been considered. It is a property of the set of curves, and as such is furnished to the set by the mind, and in Kant's view must have been a part of the mind's equipment. It becomes obvious that the study of the data of sense in any scientific way is neither more nor less than the study of the manner in which

the mind organizes these data. Science in general, and mathematics specially, is a study of the laws of thought. The laws of arithmetic are wrought into the very fiber of the mind, as well as the laws of geometry, and indeed Euclidean geometry. Mathematics is true because we find that the human mind organizes its experience in this way. These laws are not the outcome of experience, they are not derived from it, they are rather the matrices that give the fluid, unformed content of experience a definite shape. The theorems of geometry and arithmetic are true, not because they may be verified in a thousand cases or an enormous number of cases, but because they are a priori synthetic judgments that the mind is able to state from its own innate nature.

But the challenge did not delay long. Lobatchevskian geometry appeared, and was soon found to be as logical a form as Euclidean geometry. Why are not the data of sense organized in this manner rather than in the Euclidean manner? How is it possible for the mind to have three mutually inconsistent a priori schemes for the organization, the solidifying of space-data? Indeed it appears that not only do the theorems of geometry not appear to be of a purely mental character, but it is evident that they are not rigid forms of the mind, and that there is a possibility of at least some choice among them. The notions of number also as exhibited in the history of mathematics and the development of the idea of ensemble in general show an evolution which is not consistent with the idea of a priori forms of the mind, innate elements of its structure. Says Brunschvicg:[1] "No speculation on number, considered as a priori category, permits us to account for the questions of modern mathematics beginning with arithmetic

[1] *Les étapes de la philosophie mathématique*, p. 567.

itself. Not only the enunciation of the problems escapes all preassigned rule, but even the meaning of solution." It is to Kant's glory that he placed the source of mathematical verity where it belongs: in the activity of the mind. What escaped his analysis he may be pardoned for, since the doctrine of evolution was not then a scientific doctrine. It is indeed in the activity of the mind that mathematical truth originates, but not from the morphology or physiology of the mind. The mind, it is true, as Kant insisted, organizes experience, but it does this by methods that are evolutionary. It originates schemes from its own activity, and makes a choice of which of several equally valid schemes it will use. Mathematics has finally through the long development of the ages reached a vantage-point from which it is able to guarantee the freedom of the mind, even from a priori conditions.

We come, therefore, to the latest source to which mathematical verity may be ascribed, the creative activity of the mind, which ever evolves newer and higher forms of thought. The whole history of mathematics shows this constant evolution from simple notions and broad distinctions to increasingly subtle distinctions and intricacy of form. The *schema* of Kant have received a new and richer significance. For instance, one of the latest categories of mathematics appeared when the notion of functional space was born. This notion was not extracted by analysis as a sort of residue, or the fiftieth distillation of existing categories or data of the senses; it has never been claimed as an a priori notion of the human mind; it simply appeared, a new and living creature of thought. In its struggle with the data of experience the mind has had to do the best it could in many circumstances and to work its way toward freedom by a devious route, but in mathe-

matics it has come into its own at last, and is able to see clearly that it is free to create such a body of knowledge as it finds either interesting or useful for handling its problems. What the future developments in mathematics will be no one can say. It is certain, however, that the field is not a closed one. Mathematics will not be simply concerned with the minute and intensive cultivation of the fields it already knows. New notions to apply to those fields will be invented, and new fields to cultivate will be created. Its truth is dependent, not upon an empirical world, nor a transcendent reality, nor a structure of the mind, but upon an increasing power and a higher facility of the mind for devising a structure which will inclose the data of experience, and a mode of arranging these which will serve to explain them. In studying mathematics we study the constructions of the mind which relate to certain classes of entities. Since these show a progressive and stable character according to which the mind has built patiently age by age, we may conclude that this stability is a witness to a character of the mind which is an essential character, and that what it works out in the future will have the same character. Its structures are stable even when like the Ptolemaic astronomy or the corpuscular theory of light they are left standing unused. There is real truth in these systems even though they are not applied to phenomena. The source of mathematical truth is then the progressive development of the mind itself. If mind were unorganized, or chaotic, or its constructions only fleeting as the fantasies of dreams, mathematics would be impossible. If some of its creations were applied to practical life, such as four-dimensional space, it might turn out that living would be more expensive than it is, but there is no impossibility in such

applications. But granted that the mind can create and that its creations are, perforce, not self-destructive, but evolutionary, then we have a guaranty of that truth which really inheres in mathematics. Whether such truth can be made of use in our daily life is another question which we must discuss in another chapter, in the field of validity of mathematics. We may quote, as to the source of mathematical truth, Brunschvicg:[1]

> The truth of the science does not imply the existence of a transcendental reality; it is bound to the processes of verification which are immanent in the development of mathematics. It is this verification that we have believed we could uncover at the root of the constitutive notions of knowing; it is that which we have encountered at the decisive moments when the human mind saw wider horizons, as well in the book of the scribe Ahmes, who gave the proof of his calculations with fractions, as in the primary investigations of Newton and Leibniz in finding by arithmetic and algebra the results they had already obtained by the use of infinite series. Mathematical philosophy has ended its task by setting itself to follow the natural order of history, by becoming conscious of the two characters whose union is the specific mark of intelligence: indefinite capacity of progress, perpetual disquietude as to verification.

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CHAPTER XV

THE METHODS OF MATHEMATICS

We may distinguish four distinct methods by which mathematical investigation proceeds. These are not exclusive of one another, of course, but may all appear in the same piece of research, and usually would appear. The names for the four have been chosen as roughly characterizing the methods. These are: the scientific method, the intuitive method, the deductive method, and the creative method. We will consider these in sufficient detail to make clear what we mean.

1. *The scientific method.*—It is commonly supposed that mathematics has nothing to do with observation, experimentation, analysis, and generalization, the chief features of the strictly scientific method. In answer to this we may quote Sylvester:[1]

Most, if not all, of the great ideas of modern mathematics have had their origin in observation. Take, for instance, the arithmetical theory of forms, of which the foundation was laid in the diophantine theorems of Fermat, left without proof by their author, which resisted all efforts of the myriad-minded Euler to reduce to demonstration, and only yielded up their cause of being when turned over in the blowpipe flame of Gauss' transcendent genius; or the doctrine of double periodicity, which resulted from the observation of Jacobi of a purely analytical fact of transformation; of Legendre's law of reciprocity; or Sturm's theorem about the roots of equations, which, as he informed me with his own lips, stared him in the

[1] *Nature*, 1 (1869), p. 238.

face in the midst of some mechanical investigations connected (if my memory serves me right) with the motion of compound pendulums; or Huyghen's method of continued fractions characterized by Lagrange as one of the principal discoveries of that great mathematician, and to which he appears to have been led by the construction of his Planetary Automaton; or the new algebra [invariants], speaking of which one of my predecessors (Mr. Spottiswoode) has said, not without just reason and authority, from this chair, "that it reaches out and indissolubly connects each year with fresh branches of mathematics, that the theory of equations has become almost new through it, algebraic geometry transfigured in its light, that the calculus of variations, molecular physics, and mechanics [he might, if speaking at the present moment, go on to add the theory of elasticity and the development of integral calculus] have all felt its influence."

And more recently we have the remarks of Hobson[1] on the same subject:

The actual evolution of mathematical theories proceeds by a process of induction strictly analogous to the method of induction employed in building up the physical sciences; observation, comparison, classification, trial, and generalization are essential in both cases. Not only are special results, obtained independently of one another, frequently seen to be really included in some generalization, but branches of the subject which have been developed quite independently of one another are sometimes found to have connections which enable them to be synthetized in one single body of doctrine. The essential nature of mathematical thought manifests itself in the discernment of fundamental identity in the mathematical aspects of what are superficially very different domains.

One of the best examples we can find of this, as well as the other methods of mathematics, is Poincaré, whose

[1] *Nature*, 84 (1910), p. 290.

immense wideness of generalization, said Darwin, and abundance of possible applications are sometimes almost bewildering. He invented Fuchsian functions, then he found that they could be used to solve differential equations, to express the co-ordinates of algebraic curves, and to solve algebraic equations of any order. The very simple substitutions of sines and cosines or hyperbolic functions which enable us to solve quadratics and cubics were in this way generalized so that a single method—the uniformization of the variables—enables us to solve any algebraic equation, and to integrate any algebraic expression. The theory of continuous groups he applied to hypercomplex numbers and then applied hypercomplex numbers to the theory of Abelian integrals, and was able in this way to generalize the properties of the periods. He generalized the notion of Green's function, discovering the wide branch of fundamental functions and their uses. He generalized the notion of invariant to integrals over lines, surfaces, volumes, etc., and was able to reach a new point of attack on problems of dynamics. He generalized the figures of equilibrium for the heavenly bodies, discovering an infinity of new forms, and pointing out the transitions from one form to another. To state all his generalizations would take too much space and would only emphasize the great importance of the method.

There is no essential difference between generalizations of this mathematical type and those of science. It is generalization to say that projective geometry merely states the invariants of the projective group, and that elementary geometry is a collection of statements about the invariants of the group of motions. Expansions in sines and cosines are particular cases of expansions of fundamental functions in general. It is generalization to

reduce the phenomena of light first to a wave-theory, then those of light, electricity, and magnetism, to the properties of the ether. It is generalization to reduce all the laws of mechanics to the geometry of four-dimensional Lobatchevskian space. When we say natural law, we mean generalization of some kind.

Usually the process of generalization takes place by means of the various analogies present. The observation of these is necessary to generalization. But there is another mode also which leads to generalization, and that is the removal of premises in arguments, or at least of parts of premises. Much mathematical work of the present day consists in determining whether a conclusion can persist if the premises are made a little less restricted. Some element is removed from the postulates or from the defining character of the expressions, and it is then found that the conclusions still hold. For example, many theorems announced for functions of rather restricted type that are to be integrated are much more widely true if the integration is defined in Lebesgue's manner. The analogies found to exist between widely different theories enable us to see from the one theory and its developments how unsuspected developments may be made in another analogous theory. This is one of the reasons why mathematicians value even the most isolated investigations. As Whewell said: "If the Greeks had not cultivated conic sections, Kepler could not have superseded Ptolemy; if the Greeks had cultivated dynamics, Kepler might have anticipated Newton." And we may add that, if the Greeks had perceived the analogies between many theorems on conic sections, they would have invented projective geometry. If the world had seen the purely postulational character of much of geometry, long investi-

gations into the parallel axiom would have been saved, although the resulting insight into many theorems might have also disappeared.

2. *The intuitional method.*—The second method of arriving at mathematical results is by the use of the intuition. In essence this method is that of directly appreciating or seeing what is given immediately, and not as a result of deduction or other process usually called reasoning. It has different forms, whose description may be left to the psychologist. A visualist, for instance, will think in terms of pictures, diagrams, and mechanical models. We mention Faraday and his lines of force, Kelvin and his forms of ether. Out of the diagrams themselves the visualist will seize upon properties of the things he is discussing, which then can be verified by a deductive or generalizing process. The representations of analytic forms of various types by geometric figures is one of the ways in which the intuitive method is applied. A very good example is the use of the divisions of a sphere to represent the polyhedral groups, and thus to study their structure and connection with algebraic equations. Klein considered the properties of Abelian integrals by considering electric currents on closed surfaces. The consideration of the properties of sound-waves throws light on differential equations. Even Archimedes made use of known mechanical properties to assist in the calculation of areas and volumes.

The intuition is not restricted, however, to the results that may appear in some kind of visualization or physical representation of the problem. It broadens out into a profounder insight, which sees the relations that are essential, as, for instance, the insight of Lagrange when he

saw that the resolution of the algebraic equation was simply a question of functions that were invariant under the interchange of the roots. It is the insight of Riemann, who connects the deformation of surfaces and the theory of algebraic functions. It is the insight of Poincaré, who discussed the forms of the curves defined by differential equations, making an intuitive study in this way of their very intricate properties. The introduction by Hermite of continuous variables in the problems of arithmetic forms enabled him to write down at once many of the properties of the forms. The identification of functions with vectors on an infinity of unit directions and the use of such terms as orthogonal functions make intuitively evident most of the properties of integral equations. Intuition is that clear perception that enables the mathematician to keep in sight his problem and the importance of every notion that appears for the problem. It is insight of this character that enables him to identify his problem with another, to think analysis in geometric terms and geometry in analytic terms, to utilize physics to his own purposes by seeing in a physical problem exactly what he has in his analytical problem. It is for such uses as these that the development of physics is useful for the mathematician. It is insight of this kind that enables him to work intuitively in four-dimensional space, in modular space, in non-Euclidean space, in the realm of Archimedean numbers, in the region of the higher ensembles, in the corpora of algebraic numbers, in the modular forms of Kronecker, in projective differential geometry, in the functional space. It is what Klein meant when he said:[1] "Mathematics is in general at bottom the science of the self-evident." It is at the

[1] *Anwendung der Diff.- und Int.-rechnung auf geometrie*, p. 26.

root of what Pringsheim[1] meant: "A single formula contains infinitely more than all the logarithm tables on earth; for it contains the unbounded multitude of all possible thinkable cases, while any logarithm table, be it never so rich in numbers or however thick, can contain only a limited number of cases. Of the true significance and wonderful power of an analytical formula, Schopenhauer had no conception." The ability to perceive this wealth of application and richness of meaning is intuition in the higher sense. It is only by intuition of this kind that a process of logic can proceed, for the constant supervision of the process, the selection of premises, the choice of conclusions (for in the logic of relatives there are many conclusions to the same argument), the perception of the goal to be attained by the logic—all these are the work of the intuition. A fine image of Poincaré's exemplifies the matter. He compares with a sponge the final statement of a piece of mathematical investigation, which, when we find it, is fully formed and consists of a delicate lacework of silica needles. The construction of this lacework, however, was the work of a living creature, and not to be discovered merely by a study of the dead, though finished, product.

It is the intuitive method that enables mathematics to pass in the direction just opposite to that of logic, namely, from the particular to the general. It is primarily a method of discovery and often starting from a few particular cases is able to see in them theorems that are universally true. It must be accompanied by a keen power of analysis and ready perception of what is essential. It often happens that hasty generalization would lead to results that are not valid for many new cases, for the analytical power must be very keen. For instance,

[1] *Jahrb. Deutsch. Math. Ver.*, 13 (1904), p. 363.

the analysis Poincaré made of the solutions of differential equations showed that in general the integral curves wind around a limit cycle asymptotically, a result that could not have been generalized out of the few cases that permit integration in terms of elementary functions. None of these have the property in question, for the fact that they are integrable in elementary terms is due to a feature that eliminates this cycle. It would take an extremely keen intuition to perceive the importance of this feature. The method of generalization is liable to this weakness, a fact pointed out by Hadamard,[1] so that the generalized problem must be swept very carefully with the intuitive eye for characters of this kind. It is for this reason that Poincaré insists upon the great value of a qualitative study of a problem in all its aspects.

The student who desires to cultivate the intuitional method can do no better than to study the work of Poincaré, for he had a penetrating insight, and in every problem which he considered he brought out in sharp relief the essential characters. His methods of attack consist in large measure in focusing a brilliant light on the problem and examining it minutely. This method is difficult of acquisition, but should be the goal of every mathematician. Without ability of this kind, any other ability is at least badly cramped. The first-hand study of the masters of mathematics is in general the most successful method of acquiring certain skill of one's own. The object of such study should be, not so much their results, as their methods of arriving at results. In particular should be consulted Poincaré[2] and Fehr.[3]

[1] *Bibliothèque du congrès Intern. de phil.*, 3, p. 443.

[2] *Science et méthode*, p. 43.

[3] Enquête sur la méthode de travail des mathématiciens, 1908.

3. *The deductive method.*—This method has been applied so long in the history of the subject that the world has often come to the conclusion that it is the only method of mathematics. This view has been somewhat justified because most of the investigations of mathematics are published in the deductive or rigorously logical form. Often it is not at all obvious how the original investigation arrived at the results stated, and the reader is prone to wonder at the marvelous reasoning power by which an intricate piece of analysis is carried to a successful finish. In most cases it is safe to say that the results were not discovered in any such fashion, but were come upon accidentally or else by the intuitive power of the investigator. The mode of presentation often carefully removes all vestiges of the first attack. The deductive method is usually applied in the investigation as a means of verification of theorems discovered some other way, or of confirming or condemning conjectures as to the truth. The experimental method, or method of generalization in particular, is usually in need of such verification. If one were to conjecture that, when the values of a function are such that between two function-values for two given values of the argument we find as function-value every number between the two given function-values, then the function would be continuous between the two argument-values, he would be obliged to consider the consequences of this hypothesis. In case any of the logical consequences was known to be false, then the conjecture is disproved. In case only true conclusions follow, however, nothing is known as to the function's continuity. Recourse must be had in that case either to thorough intuition or else to the creative method, which is the fourth method we discuss. By the

latter in this particular case it has been proved that the function with the property mentioned need not be continuous.

The purpose of the deductive method may, therefore, be assigned as twofold: in the first place, it is the method of exposition of results; in the second place, it is the method of verification. As an expository method it is indispensable, since not every reader can be supposed to be equipped with knowledge or with ability to follow the unbroken trail by which the discoverer reached the summit of his work, but an easy road must be provided. It is to be regretted that not more of the masterful pieces of investigation have been reported in the order in which they actually proceeded, but such reports would be somewhat more voluminous than by the logical method of exposition, and would contain many reports of failures and unsuccessful methods of reaching the goal. These would be very useful to the student, but are generally considered not sufficiently elegant in form for the presentation of results. Perhaps a good example of the intuitional method is to be found in many of Sylvester's papers. Apropos of his style Noether[1] says:

> The text is permeated with associated emotional expressions, bizarre utterances, and paradoxes, and is everywhere accompanied by notes, which constitute an essential part of Sylvester's method of presentation, embodying relations, whether proximate or remote, which momentarily suggested themselves. These notes, full of inspiration and occasional flashes of genius, are the more stimulating owing to their incompleteness. . . . His reasoning moved in generalizations, was frequently influenced by analysis, and at times was guided even by mystical numerical relations. His reasoning consists less frequently of

[1] *Math. Annalen*, 50 (1898), p. 155.

pure intelligible conclusions than of inductions, or rather conjectures incited by individual observations and verifications. In this he was guided by an algebraic sense, developed through long occupation with processes of forms, and this led him luckily to general fundamental truths which in some instances remain veiled. . . . . The exponents of his essential characteristics are an intuitive talent and a faculty of invention to which we owe a series of ideas of lasting value and bearing the germs of fruitful methods.

An example of the other form of exposition is given by Hermite's work, of which Picard[1] said:

The reading of these beautiful memoirs leaves an impression of simplicity and force; no mathematician of the nineteenth century had more than Hermite the secret of these profound and hidden algebraic transformations, which, once found, seem, on the other hand, so simple. It is such algebraic skill as Lagrange no doubt had in mind when he said to Lavoisier that some day chemistry would be as easy as algebra. . . . . His courses were lithographed and were read and pondered by all the mathematicians of his day. . . . . He loved general theorems, but on condition that they could be applied to the resolution of particular questions. Not all mathematicians have in this respect the same thoughtfulness, some are satisfied with the enjoyment of announcing a beautiful general theorem, and seem to fear that they will spoil their artistic pleasure by the thought of an application to a special problem. . . . . With few exceptions his memoirs are short. The general course of the ideas is set forth, but, particularly in his early career, the presentation is synthetic, and the task of establishing the numerous intermediate theorems, whose statement alone is often given, is left to the reader.

The chief function of the logical method, however, is that of verification. In order to attach any new work

[1] *Ann. L'école normale* (3), 18 (1901), p. 1.

solidly to the structure already existing, recourse must be had to demonstration. The demonstration must start with definitions of the new terms to be used, with certain postulates regarding them, and with references somewhere to other theorems that are already known to be true. The framing of the new definitions is a part of the procedure that needs great care, since the implications of vague definitions may lead to great error. The postulates have of late years received much attention, and several mathematicians concern themselves chiefly with the production of postulate systems for the various parts of mathematics. The postulates are examined carefully as to their independence, and some attempt is made to reduce the number as low as possible. So much stress has been laid on this part of the logical presentation of mathematics that it has sometimes in recent years seemed that the chief concern of the investigator was to remove mathematics completely from the world of living thought and make it "one vast tautology" of the implications of a few definitions and a few assumptions. The logisticians in particular overemphasized this phase. However, the postulational method of presentation has its place, which is somewhat like building the foundation for an architect's design in such wise that the concrete design will be stable for all time. The postulational method, however, is impotent to produce progress or to create new branches of mathematics, or to discover new theorems. It is the mode of rigorous presentation of what has been found some other way.[1]

The chief function of symbolic logic is to further the examination of a system of definitions and postulates, so

[1] Cf. Peano, *Formulario Mathematico;* Whitehead and Russell, *Principia Mathematica.*

as to present their consequences in complete form and thus to arrive at the necessary verification of what has been discovered by generalization or by the intuition. This is necessary because generalizations from particular to general may not always be secure owing to some essential feature that is obscured or trivial in the particular case, and intuitions may not always be profound enough to see the entire structure of the problem. It might be supposed that with a most extraordinary penetration, intuitions would always be complete and accurate, but the finitude of man seems to prevent this being the case.

4. *The creative method.*—We come now to the distinctive mark of mathematical work. It is at the same time a method whose nature is spontaneous and creative, and in consequence free from rules of procedure and difficult to characterize. We may begin, however, with the problem of generalization. There are two kinds of generalization: one, the scientific type, already considered, which extends known theorems relating to a certain domain to a wider domain—it consists sometimes in the restatement of a theorem so that in the new form it will apply to a wider domain—but the second kind, the mathematical type essentially, is the one we are now discussing, which consists in the actual creation of new entities and their study. We have found abundant examples in the preceding pages, and we considered the mark of mathematics throughout the centuries to be this ability to create new things. Examples are the irrational numbers, negative numbers, imaginary and hypercomplex numbers, Kummer's ideal numbers, Minkowski's geometric numbers, algebraic fields, fields in general. On the geometric side we found that the non-Euclidean

geometries, non-Archimedean continuity, transfinite numbers, hyperspaces of all types, imaginary space, functional space, are good examples of what has actually been created. This is the stamp of the great mathematician par excellence, that he creates a new set of entities. These entities arise generally as the demands for solutions of equations or propositions of some type or other necessitate a wider domain. The imaginary was created to make the solutions of quadratics always possible. Elliptic functions were created to make the integration of square roots of cubics and quartics always possible. The Fuchsian functions and related functions were invented to enable algebraic equations to be uniformized and to render their integration possible. Ideal numbers were created to enable the properties of integers to be extended to other numbers. Abstract fields were invented to furnish a domain in which there are no limiting processes.

Another function of the creative method is to invent cases which will show that some proposition has a limited range of validity, or that some definition needs to be further divided. A conspicuous case was the invention by Weierstrass of a function that was continuous, but had no derivative. The mere existence of such a function introduced radical changes in the definitions of functions and the criteria that were applied in certain cases. It showed that continuity was a separable property and could be resolved into several kinds of continuity. The function invented by Darboux to show that, although it took every intermediate value between two given values in passing from $x=a$ to $x=b$, it yet was discontinuous, showed, again, that one of the properties of continuity was not a sufficient property to define continuity. It is true

that creation of this critical character is not so fundamental as that of a synthetic character, but it is in the end necessary and extremely useful. Other cases might be cited, such as the curves that fill up an area, the Jordan curves, the monogenic non-analytic functions of Borel, etc. These are no more artificial than were in their day the negative and the imaginary. Indeed, the time may come when the demands of physics may make it necessary to consider the path of an electron to be a continuous non-differentiable curve, and the Borel function may become a necessity to explain the fine-grained character of matter. "To the evolution of Physics should correspond an evolution in Mathematics, which, of course, without abandoning the classic and well-tried theories, should develop, however, with the results of experiment in view."[1]

The origin of these creations is a most interesting question for the psychologist and is buried in the mysterious depths of the mind. An interesting account of it is given by Poincaré in a description of some of his own creations, to be found in his book, *Science et méthode*. His conclusion may be stated briefly thus: the mind is in a state of evolution of new ideas and new mental forms, somewhat continuously. Of those that come to the front some will have a certain relation of harmony and fitness for the problem at hand, which secures for them keen attention. They may turn out to be just what is wanted, sometimes they may turn out to be unfit or even contradictory. There seems little to add to this statement, for it pretty accurately describes what every reflective mathematician has observed in his own mental activity. A little emphasis may be laid, however, on the

[1] Borel, Lecture at Rice Institute, 1912; *Introduction géométrique à quelques théories physiques*, pp. 126–137.

significance of the fact that sometimes the newborn notions are contradictory to the known theorems, because this fact shows conclusively that the mind is not impelled to its acts by a blind causality. In that case the new forms would have to be always consistent. This faculty is analogous to that possessed by the artist. Indeed, many have noted the numerous relations of mathematics to the arts that create the beautiful. Sylvester[1] said: "It seems to me that the whole of aesthetic (so far as at present revealed) may be regarded as a scheme having four centers, which may be treated as the four apices of a tetrahedron, namely, Epic, Music, Plastic, Mathematic." Poincaré was specially the advocate of the aesthetic character of mathematics, and reference may be made to his many essays. Many others have mentioned the fact in their addresses. The cultivation of the aesthetic sensitiveness ought, therefore, to assist the creative ability of the mind.

Poincaré points out that these flashes of inspiration usually follow long and intense attention to a problem. That is, one must endeavor to generalize, to turn the searchlight of intuition on the problem, to deduce from every phase of it all the consequences that follow, and then he must trust to the spontaneity of the mind some day to furnish the newborn creature that is engendered by these processes. The process of maturing the conception may even take years. This fourth method is the culmination, the crown, of the others and of the acquisitions of the mathematical student. He must read widely, scrutinize intently, reflect profoundly, and watch for the advent of the new creatures resulting. If he is of a philosophic

[1] *Collected Papers*, 3, p. 123.

turn, he will have the satisfaction of knowing that he is able to see knowledge in the process of creation, and that of all reality he has the most secure. He will know that the flowers of thought whose growth and bloom he superintends are immortelles and the infinite seasons of the ages will see them in everlasting fragrance and beauty.

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CHAPTER XVI

VALIDITY OF MATHEMATICS

We have surveyed the whole of mathematics, finding it to be a constantly growing creation of the intellect, constructed primarily for its own sake. The mathematician builds because he enjoys the building, and the fascination of his creation is the impetus that keeps him creating. It is not the usefulness of what he creates, but the innate beauty of it that he is forever thirsting for. Poincaré,[1] the subtlest of the mathematical philosophers, said:

> The scientist does not study nature because it is useful; he studies it because it pleases him, and it pleases him because it is beautiful. Were nature not beautiful, it would not be worth knowing, life would not be worth living. I do not mean here, of course, that beauty which impresses the senses, the beauty of qualities and appearances; not that I despise it—far from it; but that has nought to do with science; I mean that subtler beauty of the harmonious order of the parts which pure intellect perceives. This it is which gives a body, a skeleton as it were, to the fleeting appearances that charm the senses, and without this support the beauty of these fugitive dreams would be but imperfect, because it would be unstable and evanescent. On the contrary, intellectual beauty is self-sufficient, and for its sake, rather than for the good of humanity, does the scientist condemn himself to long and tedious labors.

The Greeks studied conic sections two thousand years before they were of use to anyone at all, and the imaginary and complex functions were developed long before they were of use to the wireless telegrapher. Nevertheless,

[1] *Science et méthode*, p. 15.

the tree Yggdrasil has its roots in the earth and thence draws sustenance for its growth. As Poincaré and Borel point out, many of the notions of mathematics had their origin in the demands of physics for a scheme by which it could think the material world—such notions as continuity, derivative, integral, differential equation, vector calculus, and the integral and integro-differential equation. Equally the learning of mankind in its efforts to understand its own whence, why, and whither has furnished sustenance for another root of the trunk. Pythagoras, Plato, Leibniz, Kant, Poincaré—merely to mention these names brings to mind the debt of mathematics to the philosophic thought of the centuries. The search for that in life which was definitive, for freedom of the intellect, for the unity and harmony of the spirit as well as of nature—all these have contributed to the sustenance of the trunk, even if they could not be part of the tree. And the other root was the accumulated learning of the past. We are today heirs of the whole past in mathematics. Nothing is wasted, nothing is dissipated, but the wealth, the flashing gems of learning, which are the reward of painful toil of men long since dead, are ours today, a capital which enables us to advance the faster and to increase the riches all the more. With such a source of power we must then inquire where the mathematician will find a valid domain in which to justify to the rest of the world his right to exist. What is its fruitage either in bloom or in mature fruit for the sustenance of the nations? What storms will its foliage protect from, and what distant peaks with their glistening slopes are visible from its lofty summit? Since it rears its head so proudly into the rarefied upper atmosphere where only the privileged few can ever go, what can it bring down for the inspiration

and the refreshment of man? These will be our closing themes.

We may pass over the service that mathematics renders to the applied sciences, such as engineering. Its obvious importance is plain to everyone. Its chief service to science is the construction of definite models for the conceptions and theories of science. Says Milhaud:[1]

From the remotest times to our own day the good fortune of mathematical speculations continues. Mathematicians have made unimaginable progress in abstraction; the elements they define in ceaseless flow carry them farther and farther, not only from the material world, but from all concrete foundation; there is less and less of things, more and more of relations, functions, ensembles, groupings, even modes of groupings; and if there is still a language whose words have concrete significance, like space, points, lines, planes, it is only the remains of an old illusion, for the space is n-dimensional, the points, lines, and planes are as much imaginary as real, at infinity as well as at a finite distance; the functions may or may not be expressible in a finite number of symbols, they may be continuous or discontinuous, and can escape all imagery, any representation. Yet, however high mathematicians seem to be carried in their reveries above all sensible reality, all the symbols that they create and which seem naturally and spontaneously to evoke their kind by a spiritual need, find or will find a place of application—at first, doubtless, like the parabola of areas, to questions still theoretical, for example, to some transformation useful for certain analytical expressions, then sooner or later to the solution of some difficulty in mechanics, physics, or astronomy.

These are the two contradictory characters which make up the apparent miracle of mathematical thought; spontaneity in the flight of the mind, which, foreign to all utilitarian pre-

[1] *Nouvelles études sur l'historie de la pensée*, p. 30.

occupation, soars higher and higher in its abstracted musings—and incessant progress in the knowledge of the physical world by the utilization soon or late of the symbols created in this way.

We are indeed witnesses at the present time to the interaction of these characters. The notion of continuity in physics led to the invention of the infinitesimal calculus, and this led to the physics of continuous media, the classic mathematical physics, whose equations appear as partial differentials. Liquids and gases and the hypothetical ether were conceived as media which were continuous, and even expressions such as the potential function, the energy function, the dissipation function, and others in many variables appear, with the implication from their partial differentials that they are functions belonging to a continuous space or continuous medium. The tendency along this line has been to reduce these problems to problems in the calculus of variations, all the expressions and laws being reduced to mere consequences of the variation of some definite integral. We find among the latest developments of this type the equations of elastic equilibrium and the theory of vibrations of elastic bodies reducible to problems in the determination of a minimum for certain definite integrals. Here the theory of waves in continuous media finds a concrete application, and some connection is made with the discontinuities that may enter. The partial differential equations of this theory have been extensively studied and applied to all the problems of a simple character at least. Modern developments in mathematics have also been applied to the same problems, and in the use of fundamental functions and integral equations we are able to carry the solutions over a much wider range.

At the same time the total differential equation has had its influence in developing physics. The movement of a single particle has been considered to be a continuous entity, and the total differential equation enters in an important rôle. If we consider a system of particles, the number of differential equations increases. Probably the most celebrated problem of this type is that of N bodies in astronomy, which has been studied a long time and has reached a definitive solution only quite recently. If a body is supposed to be made up of small particles, approximately points, and these are supposed to move slightly from their positions, we arrive at a discrete theory of elasticity. The movements in the simplest case are periodic, the periods determining the specific constants of the body. If, now, we suppose the number of particles to approach infinity, we may substitute for the total differential equations which become infinitely numerous, a finite number of partial differential equations, and thus come back to the previous system of equations which suppose a continuous medium.

More modern developments in physics have led to atomistic conceptions of matter, electricity, and energy. The Brownian movement of atoms, the electrons in various rays, and the quanta of the radiation theories are centers of the new physical theories that are in the making. For these it seems a mathematics of the totally discontinuous is necessary. The differential equation gives place to difference equations, the definite integral to the infinite sum, the analytic function to the monogenic function. The speculations of mathematicians on the ensemble, the Lebesgue integral, the totally discontinuous, are finding their sphere of application. Not only this, but the kinetic theory of gases has demanded functions of a

very large number of variables, and the properties of bodies in space of n dimensions are thus becoming of use. N dimensional ellipsoids, where N is very large, approaching infinity we may say, have certain properties which are useful in these connections.

Physics has not only reinstated the once discarded action at a distance, which disappeared from all the theories of continuous media, but has also introduced the more recent notion of action at a distance in time. The procedure of a phenomenon may no longer depend solely upon the state immediately preceding, but may also depend upon states at some distance remote in time. What will take place tomorrow depends, not only upon what takes place today, but also upon what took place day before yesterday. In some phenomena the past is able to reach into the future and affect it as well as the present. We find, however, that mathematics is building a system of notions that are applicable here, and the functions of lines and planes and other configurations, together with the integro-differential equation, recently discussed by Volterra, are able to handle these problems. We might add that if the day comes when there are phenomena in physics, like the lines of the spectra of the elements, which can be stated in laws that involve integers, even the abstractions of the theory of numbers will be applied to the advance of physical knowledge. The domain of validity of mathematics in this direction would seem to be all natural phenomena whose model corresponds faithfully to the constructions of some part of mathematics. Whether all science can be framed according to models of this kind is for the future to say.

The development of mathematics has profoundly influenced philosophy. We need but mention Pythagoras,

Plato, Descartes, Spinoza, Leibniz, Kant, Comte, and Russell, in order to call to mind philosophers whose systems were controlled largely by their views of mathematics. There have been also mathematicians who have been at the same time philosophers, and whose criticisms have largely influenced existing systems. The existence indeed of mathematics, its evergreen growth, and its constant success in creating a body of knowledge whose value is universally admitted, are a challenge to the philosopher to do as much, and at the same time an encouragement to him to persist in his search for the explanation of things as they are. There is at the present time an increasing trend toward each other of the two disciplines. The philosopher is confronted, too, with the added difficulty that he cannot hope to have a complete system unless he accounts for the existence of mathematics and assigns a value to it in human economy, and in order to do this he must perforce learn some mathematics. He must know what the mathematician has found out for himself about his own science, and the significance of what he has found out for the rest of the theory of knowledge. The irrational wrecked the Pythagorean school, the universal wrecked the Platonic school, the reality of mathematical constructions wrecked Cartesianism, the ideality in mathematics wrecked Leibnizianism, the arbitrary constructions of mathematics wrecked Kant's philosophy, and the scientific value of them wrecked Comte's positivism, the free creation in mathematics wrecked Russell's logistic and answers Bergson's criticisms of mathematics while it substantiates his fundamental contention. The searching analysis the mathematician has made of his own conceptions has not only illuminated them, but at the same time has cleared

the fog away from some of the philosophical notions. The universe cannot be constructed by mere thinking, mathematics and other thinking are not the result of a universal characteristic, the intuition has been reinstated, and at the same time conditioned in its action, the real source of verity in mathematics has been exhibited. Says Brunschvicg:[1]

> The mathematical intellectualism is henceforth a positive doctrine, but one that inverts the usual doctrines of positivism: in place of originating progress in order, dynamics in statics, its goal is to make logical order the product of intellectual progress. The science of the future is not enwombed, as Comte would have had it, as Kant had wished it, in the forms of the science already existing; the structure of these forms reveals an original dynamism whose onward sweep is prolonged by the synthetic generation of more and more complicated forms. No speculation on number considered as a category a priori enables one to account for the questions set by modern mathematics . . . . space affirms only the possibility of applying to a multiplicity of any elements whatever, relations whose type the intellect does not undertake to determine in advance, but, on the contrary, it asserts their existence and nourishes their unlimited development.

These things the philosopher must learn along with his apprehension of modern science and all it, too, has to say about the world, knowledge, and truth. "The consideration of mathematics is at the base of knowledge of the mind as it is at the base of the natural sciences, and for the same reason: the free and fertile work of thought dates from that epoch when mathematics brought to man the true norm of truth."[2]

[1] *Les étapes de la philosophie mathématique*, pp. 567–568.

[2] Brunschvicg, p. 577.

Finally we find a domain for the validity of mathemathics in a region that might seem at first remote indeed. But nevertheless the truth in mathematics, a free creation of the imagination incarnated in forms of the reason, guarantees the truth of other free creations of the imagination when they are set forth in the realities of life. Poetry, music, painting, sculpture, architecture—may we call them the other fine arts?—create the beautiful and give expression to the longings and hopes of man. But they have been told for centuries that these were but dreams, visions of that which did not exist, sad to say, fictions that one could but view for awhile, then, with a sigh, return to cold reality. Mathematics vindicates the right of all these to stand in the front rank of the pioneers that search the real truth and find it crystallized forever in brilliant gems. The mathematician is fascinated with the marvelous beauty of the forms he constructs, and in their beauty he finds everlasting truth. The scientist studies nature for the same reason, and in its harmonies finds also everlasting truth. But the nature he studies is the creature of his own construction. His conceptions and theories and scientific systems he really builds himself. So, too, the artist sees beauty and constructs imperishable forms which also have everlasting truth. Many mathematicians have borne witness to the element of beauty in mathematics: Poincaré, high priest of beauty in mathematics and science, Sylvester, who wrote rhapsodies in the midst of his mathematical memoirs, Pringsheim, Kummer, Kronecker, Helmholtz, Bôcher, B. Peirce, Russell, Hobson, Picard, Hadamard—why prolong the list? And because mathematics contains truth, it extends its validity to the whole domain of art and the creatures of the constructive imagination. Because it contains freedom, it guarantees

freedom to the whole realm of art. Because it is not primarily utilitarian, it validates the joy of imagination for the pure pleasure of imagination.

"Not in the ground of need, not in bent and painful toil, but in the deep-centred play-instinct of the world, in the joyous mood of the eternal Being, which is always young, science has her origin and root; and her spirit, which is the spirit of genius in moments of elevation, is but a sublimated form of play, the austere and lofty analogue of the kitten playing with the entangled skein or of the eaglet sporting with the mountain winds."[1]

REFERENCES

Keyser, *The Human Worth of Rigorous Thinking*, 1916.

[1] Keyser, *Mathematics*, Columbia University Lectures, 1907.

STRUCTURE OF MATHEMATICS

				Objects	Form
Ranges	Static Mathematics	Chaotic	Arithmetic	Integers Rationals Irrationals Ensembles NUMBER	Integers Ensemble theory Literal arithmetic Expansions Complexes Point geometries Functional spaces
			Multiplexes	Points Lines Surfaces Varieties Higher elements SPACE	Point space of two o more dimensions Line geometry Surface geometry Absolute geometry Higher elements
		Ordered	Tactic	Arrangements Configurations Constellations STRUCTURE	Combinatory analysi Stereochemistry Theory of symbols
			Logistic	Concepts Relatives PROPOSITIONS	Foundations Postulational system Calculus of classes, o of relatives
Transitions	Dynamic Mathematics	Chaotic	Operators	Substitutions Transformations Groups MUTATION	Theory of finite or infinite groups Calculus of operation
			Algebra	Negatives Imaginaries Hypernumbers QUALITY	Linear associative algebras Functional transfor- mations
		Ordered	Processes	Routes Displacements Combinations ACTION	Composite actions Actional structures
			Inferences	Transpositions Syllogisms Implications DEDUCTION	Syllogisms Calculus of deductio

Invariancy	Functionality	Ideality
Congruences Arithmetic forms Arithmetic invariants Modular geometry	Arithmetic functions Algebraic functions Functions of real variable Infinitesimal calculus General analysis	Arithmetic ideals Galois ideals Higher number theory Difference equations Picard-Vessiot theory Differential equations
Geometric invariants Algebraic invariants Symmetric and alternating forms Modular systems	Real functions of N variables Vector fields Functions of lines Partial derivatives Differential geometry	Systems of differential equations Mathematical physics
Transitivity Primitivity Stable systems Irreducibles	Functions of arrangements, configurations, constellations, and structures	Ideal elements of construction Mathematical chemistry
Equivalent systems Logical invariants	Classes functions of classes Functions of relatives Implication	Ideal classes or relatives Classificatory schemes
Projective geometry Inversion geometry Differential and integral invariants Analysis situs	Geometric transformations Homomorphisms Transmutations	Automorphic functions Functional equations Calculus of variations Functional analysis Integral equations
Invariant equations Invariants of expressions	Functions of complex and hypercomplex variables General function theory	Hyperideals
Equivalents of actions Invariants of processes	Processes dependent on processes Functions of action	Ideal processes
Laws of inference Equivalent deductions Invariants	Functions of inferences	Ideal entities that satisfy inferences Scientific theories

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